

Norms on Quotient Spaces of The 2-Inner Product Space

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Abstract. This paper discussed about construction of some quotients spaces of the 2-inner product spaces. On those quotient spaces, we defined an inner product with respect to a linear independent set. These inner products was derived from the n -inner product. We then defined a norm which induced by the inner product in these quotient spaces.

Keywords: n -inner product spaces, norm, quotient space.

1 Introduction

The concept of the inner product space

In 1960's S. Gähler initially introduced the concept of 2-normed space [1-4] which is a generalization of the concept of norm space. Let X be a real vector space of dimension $d \geq 2$. A real-valued function $\|\cdot, \cdot\|$ on $X \times X$ that satisfies the following conditions:

- N1. $\|x, y\| = 0$ if and only if x and y are linearly dependent, for all $x, y \in X$;
- N2. $\|x, y\| = \|y, x\|$, for all $x, y \in X$;
- N3. $\|\alpha x, y\| = |\alpha| \|x, y\|$ for all $\alpha \in \mathbb{R}$ and $x, y \in X$;
- N4. $\|x, y + z\| \leq \|x, y\| + \|x, z\|$; for all $x, y, z \in X$;

is called a 2-norm on X , and the pair of $(X, \|\cdot, \cdot\|)$ is called a 2-normed spaces.

For example, if $(X, \langle \cdot, \cdot \rangle)$ is an inner product space then the following function

$$\|x, y\| = \left| \begin{array}{cc} \langle x, x \rangle & \langle x, y \rangle \\ \langle y, x \rangle & \langle y, y \rangle \end{array} \right|^{\frac{1}{2}}$$

is a 2-norm in X . We can check that the above function satisfies (N1 – N4). Geometrically, the value of $\|x, y\|$ represents the area spanned by x and y . We called it standard 2-norm on X .

Moreover, one can see that from (N3) and (N4) we have

$$\|x, y\| \geq 0, \text{ for any } x, y \in X,$$

which means the 2-norm is nonnegative. On 2-normed space, we also have

$$\|x, \alpha x + y\| = \|x, y\|, \text{ for any } \alpha \in \mathbb{R} \text{ and } x, y \in X.$$

A concept which is related to the 2-normed space is the 2-inner product space. This is a generalization of the concept of inner product space. Concept of 2-normed space and 2-inner product space it has been developed extensively with various results by many researchers, (see for instance [5-11]).

Let X be a real vector space and $\dim(X) = d \geq 3$. A function $\langle \cdot, \cdot | \cdot \rangle: X \times X \times X \rightarrow \mathbb{R}$ which satisfies the following conditions

- I1. $\langle x, x|z \rangle \geq 0$ and $\langle x, x|z \rangle = 0$ if and only if x and z are linearly dependent;
- I2. $\langle x, x|z \rangle = \langle z, z|x \rangle$, for any $x, z \in X$;
- I3. $\langle x, y|z \rangle = \langle y, x|z \rangle$, for any $x, y, z \in X$;
- I4. $\langle \alpha x, y|z \rangle = \alpha \langle x, y|z \rangle$, for any $\alpha \in \mathbb{R}$ and $x, y, z \in X$;
- I5. $\langle x_1 + x_2, y|z \rangle = \langle x_1, y|z \rangle + \langle x_2, y|z \rangle$;

is called a 2-inner product on X , while the pair $(X, \langle \cdot, \cdot | \cdot \rangle)$ is called a 2-inner product space.

We can derive a 2-norm from a 2-inner product by define

$$\|x, z\| = \langle x, x|z \rangle^{\frac{1}{2}}. \quad (1)$$

We say that the 2-norm in (1) is induced by a 2-inner product.

Furthermore, we will construct some quotient spaces in a 2-inner product space and define an inner product in each quotient space. If X be a vector space, then we can construct some quotient spaces of X . These quotient spaces can be constructed with respect to some equivalent relations. Moreover, we can define an inner product in X whenever it is possible.

Let \sim be an equivalence relation on X . For any $x \in X$, the set of all elements equivalent to x is denoted by

$$\bar{x} := \{y \in X : y \sim x\},$$

is called the equivalence class of x .

Let X be a vector space over F , $x \in X$, and $V \subseteq X$. The set $\bar{x} = x + V = \{x + v : v \in V\}$ is called a coset of V in X , x is called a coset representative for $x + V$. Moreover, the set of all cosets of V on X is denoted by

$$X/V = \{\bar{x} : x \in X\}.$$

This set is called the quotient of X modulo V . [12]

An addition and scalar multiplication operation on X/V is defined by $\bar{x} + \bar{y} = \overline{x + y}$ and $\alpha\bar{x} = \overline{\alpha x}$ respectively, for any $x, y \in X/V$ and $\alpha \in F$.

2 Results and Discussion

Now, we will construct some quotient spaces in a 2-inner product space. Let $(X,)$ be a 2-inner product space and $Y = \{y_1, y_2\}$ be a linear independent set on X . Define a set of X generated by $\{y_1\}$

$$Y_1 = \text{span}\{y_1\} = \{\alpha y_1 : \alpha \in \mathbb{R}\}.$$

It is easy to see that Y_1 is a subspace. For any $x \in X$, the corresponding coset of Y_1 on X is

$$\bar{x} = \{x + \alpha y_1 : \alpha \in \mathbb{R}\}.$$

Then we have $\bar{0} = \text{span}\{y_1\} = Y_1$, and if $\bar{x} = \bar{y}$ then $x - y \in Y_1$. Further, we define the quotient space of X as

$$X_1^* = X/Y_1 = \{\bar{x} : x \in X\}. \quad (2)$$

The addition and scalar multiplication also apply on X_1^* , those are

1. $\bar{x} + \bar{y} = \overline{x + y}$, for for any $x, y \in X/Y_1$;
2. $\alpha\bar{x} = \overline{\alpha x}$ respectively, for any $x \in X/Y_1$ and $\alpha \in \mathbb{R}$.

Next, on X_1^* we define a function $\langle \cdot, \cdot \rangle_1 : X_1^* \rightarrow \mathbb{R}$ defined by

$$\langle \bar{x}, \bar{y} \rangle_1 = \langle x, y | y_1 \rangle \quad (3)$$

We can see easily that the function defined on (3) is well defined. This function defines an inner product on X_1^* , so that the pair $(X_1^*, \langle \cdot, \cdot \rangle_1)$ is an inner product space as stated in the following theorem.

Theorem 2.1. Let $(X, \langle \cdot, \cdot | \cdot \rangle)$ be a 2-inner product space and $Y = \{y_1, y_2\}$ is a linearly independent set in X . Then $(X_1^*, \langle \cdot, \cdot \rangle_1)$ is an inner product space, where $\langle \cdot, \cdot \rangle_1$ is a function in (3).

Proof.

To prove the above theorem we have to prove that the function we defined in (2) is an inner product on X_1^* . Recall that

$$\langle \bar{x}, \bar{y} \rangle_1 = \langle x, y | y_1 \rangle,$$

then using the properties of 2-inner product (N1) we have

$$\langle \bar{x}, \bar{x} \rangle_1 = \langle x, x | y_1 \rangle \geq 0.$$

If $\langle \bar{x}, \bar{x} \rangle_1 = 0$, then $\langle x, x | y_1 \rangle = 0$. This means x and y_1 are linearly dependent. We can write $x = \alpha y_1$, for an $\alpha \in \mathbb{R}$ or $x \in \text{span}\{y_1\} = Y_1$, which means $\bar{x} = \bar{0}$. Conversely, if $\bar{x} = \bar{0}$ then it is obvious that $\langle \bar{x}, \bar{x} \rangle_1 = \langle \bar{0}, \bar{0} \rangle_1 = \langle 0, 0 | y_1 \rangle = 0$.

Next, using (N3) and (N4) we have for any $\alpha \in \mathbb{R}$ and $x, y \in X$, we have

$$\langle \bar{x}, \bar{y} \rangle_1 = \langle x, y | y_1 \rangle = \langle x, y | y_1 \rangle = \langle \bar{y}, \bar{x} \rangle_1$$

and

$$\langle \alpha \bar{x}, \bar{y} \rangle_1 = \alpha \langle x, y | y_1 \rangle = \alpha \langle \bar{x}, \bar{y} \rangle_1$$

Moreover, using (N5) we have for any $x_1, x_2 \in X$

$$\langle \overline{x_1 + x_2}, \bar{y} \rangle_1 = \langle x_1 + x_2, y | z \rangle = \langle x_1, y | z \rangle + \langle x_2, y | z \rangle = \langle \bar{x}_1, \bar{y} \rangle_1 + \langle \bar{x}_2, \bar{y} \rangle_1.$$

These prove that the function on (3) is an inner product space on X_1^* , which means $(X_1^*, \langle \cdot, \cdot \rangle_1)$ is an inner product space. \square

Recall that if $(X, \langle \cdot, \cdot \rangle)$ is an inner product space then the following formula

$$\|x\| = \langle x, x \rangle^{\frac{1}{2}},$$

defines a norm on X , where $x \in X$. By this fact, we can define a norm on X_1^* .

Let $(X, \langle \cdot, \cdot | \cdot \rangle)$ be a 2-inner product space and $Y = \{y_1, y_2\}$ is a linearly independent set in X . A function $\|\cdot\|_1: X_1^* \rightarrow \mathbb{R}$ which defined by

$$\|x\|_1 = \langle x, x \rangle_1^{\frac{1}{2}} \tag{4}$$

is a norm on X_1^* .

By (3) and (1) we can see that

$$\langle x, x \rangle_1^{\frac{1}{2}} = \langle x, x | y_1 \rangle_1^{\frac{1}{2}} = \|x, y_1\|.$$

This means The norm $\|\cdot\|_1$ defined in (4) can be seen as 2-norm on X with respect to y_1 .

Furthermore, we will construct another quotient space in a 2-inner product space. Let $(X, \langle \cdot, \cdot | \cdot \rangle)$ be a 2-inner product space and $Y = \{y_1, y_2\}$ be a linear independent set on X . For constructing the previous quotient space we considered $y_1 \in Y$. Now, we consider $y_2 \in Y$ and use the same construction to define another quotient space. Define a set of X generated by $\{y_2\}$

$$Y_2 = \text{span}\{y_2\} = \{\alpha y_2 : \alpha \in \mathbb{R}\}.$$

The set Y_1 is a subspace and for any $x \in X$, the corresponding coset of Y_2 on X is

$$\bar{x} = \{x + \alpha y_2 : \alpha \in \mathbb{R}\}.$$

Then we have $\bar{0} = \text{span}\{y_2\} = Y_2$, and if $\bar{x} = \bar{y}$ then $x - y \in Y_2$. Further, we define the quotient space of X as

$$X_2^* = X/Y_2 = \{\bar{x} : x \in X\}. \quad (5)$$

The addition and scalar multiplication also apply on X_2^* .

On X_1^* we define a function $\langle \cdot, \cdot \rangle_2 : X_2^* \rightarrow \mathbb{R}$ defined by

$$\langle \bar{x}, \bar{y} \rangle_2 = \langle x, y | y_2 \rangle. \quad (6)$$

This function defines an inner product on X_2^* , so that the pair $(X_2^*, \langle \cdot, \cdot \rangle_2)$ is an inner product space as stated in the following theorem.

Theorem 2.2. Let $(X, \langle \cdot, \cdot | \cdot \rangle)$ be a 2-inner product space and $Y = \{y_1, y_2\}$ is a linearly independent set in X . Then $(X_2^*, \langle \cdot, \cdot \rangle_2)$ is an inner product space, where $\langle \cdot, \cdot \rangle_2$ is a function in (6).

Proof.

The proof is analogue with the proof of Theorem 2.1. □

As an analogue result, we also have the following formula

$$\|x\|_2 = \langle x, x \rangle_2^{\frac{1}{2}}; x \in X \quad (7)$$

defines a norm on X_1^* .

By (6) and (1) we can see that

$$\langle x, x \rangle_2^{\frac{1}{2}} = \langle x, x | y_2 \rangle^{\frac{1}{2}} = \|x, y_2\|.$$

This means The norm $\|\cdot\|_2$ defined in (4) can be seen as 2-norm on X with respect to y_2 .

On [13], Gunawan derived an inner product on an n -inner product space. The inner product is derived from the n -inner product. Let $(X, \langle \cdot, \cdot | \cdot, \dots, \cdot \rangle)$ and $Y = \{y_1, \dots, y_n\}$ be a linearly independent set. Gunawan derived an inner product on X , defined by

$$\langle x, y \rangle = \sum_{\{i_1, \dots, i_n\} \subseteq \{1, \dots, n\}} \langle x, y | y_{i_1}, \dots, y_{i_n} \rangle.$$

For $n = 2$, the above formula can be written as

$$\langle x, y \rangle = \langle x, y | y_1 \rangle + \langle x, y | y_2 \rangle \quad (8)$$

As we can see, each term of right hand side of equation (8) is the formula of inner product on two quotient spaces we constructed earlier.

$$\langle x, y \rangle = \langle x, y \rangle_1 + \langle x, y \rangle_2 \quad (9)$$

As a consequence, using both of inner products we constructed earlier we can investigate some aspects the inner products of both space. Moreover, the norm induced by (9) can be written as

$$\|x\| = \langle x, y \rangle^{\frac{1}{2}} = (\langle x, y \rangle_1^2 + \langle x, y \rangle_2^2)^{\frac{1}{2}} = (\|x\|_1^2 + \|x\|_2^2)^{\frac{1}{2}}. \quad (10)$$

On the above norm, we also can see the induced norms on two quotient spaces we constructed on the right hand side of equation (10).

Furthermore, we can use these inner products or the induced norms to observe some aspects on an n -inner product space.

3 References

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