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Rainbow Connection on $Amal(F_n, xz, m)$ Graphs and $Amal(O_n, xz, m)$ Graphs

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Abstract. Coloring graph is giving a color to a set of vertices and a set of edges on a graph. The condition for coloring a graph is that each color is different for each neighboring member graph. Coloring graph can be done by mapping a different color to each vertex or edge. Rainbow coloring is a type of rainbow connected with coloring edge. It ensures that every graph G has a rainbow path. A rainbow path is a path in a graph where no two vertices have the same color. The minimum number of colors in a rainbow connected graph is called the rainbow connection number denoted by $rc(G)$. The graphs used in this study are the $Amal(F_n, xz, m)$ graph and the $Amal(O_n, xz, m)$ graph.

Keywords: *$Amal(F_n, xz, m)$ graph; $Amal(O_n, xz, m)$ graph; coloring graph; edge amalgamation; rainbow connection number.*

1 Introduction

A graph G is a pair of VG non-empty finite sets and EG sets that can be empty with VG as a point, while EG is the side between two points [1]. Lots of points (*vertex*) on the graph G called the order is notated with $|V(G)|$ while the number of sides (*edges*) on the graph G called the size is denoted with $|E(G)|$. In graphs G, there are also degrees on many adjacent sides with point x on the graph G. The largest (maximum) degree at the graph G is denoted by $\Delta(G)$ and the smallest (minimum) degree of the graph $G$is denoted by $\delta(G)$. The longest distance between any two points on a graph G is called the diameter denoted with $diam(G)$.

The coloring rainbow is one of the coloring part sides in the graph G. For example, a graph G is a connected graph with edge coloring $c: E(G) \to \{1,2,3,\dots,k\}$, provided that k is the part of a natural number with neighboring side can own the same color. Trajectory rainbow in graph G is formed if two points in the graph do not have paths with same color. Coloring rainbow is also called edge coloring in a G rainbow-connected graph. Coloring rainbow has a minimum color called rainbow connection number. The rainbow connection number on graph G can be denoted by $rc(G)$ [2].

Chartrand et al. [3] research about rainbow connection on several types of graphs, including cycle graph, complete graph, trees graph, and wheel graph. Kemnitz et al. [4] also study about graphs with two rainbow connection numbers. Besides that, Li et al. [5] found rainbow connection on the graph with a diameter of 2. Syafrizal et al. [6] study about rainbow connection on several graphs, there are gear graph, book graph, fan graph, and sun graph.

With two points, u and v, from G, a *rainbow geodesic*$-(u,v)$ in G is *rainbow path*$-(u,v)$ with $d(u,v)$ is the distance between u and v $(\text{the length of the shortest path} - (u,v)\ in\ G)$. The graph G is strongly rainbow connected if G has a *rainbow geodesic*$-(u,v)$ for every two points $u, v \in G$. In this case, coloring c is said to be a strong rainbow coloring from $G(src(\mathrm{G}))$. The minimum k contained in the coloring $c: E(G) \rightarrow \{1,2,3,\ldots,k\}$ of G such that an edge G is strongly rainbow-connected with strong rainbow connection number, $src(G)$ from G [3].

Operation graph is a method to get the form arrangement of new graph that symbolized with $Amal(G, xz, m)$. The graph is operated on originates from the base graph and then operated on with the copy results. In the previous study that related to the operation of amalgamation is Fitriani and Salman [7]. Their research found results with a rainbow connection number on the operation of amalgamation point that false only with the fan graph. In this study, this operation graph is used for operation of amalgamation. On the graph results from operation amalgamation with m copy, the selected one fixed side or terminal side on each m graph copy and then paste it into one side on the fixed side. The type of graph used in this study is $Amal(F_n, xz, m)$ graphs and $Amal(O_n, xz, m)$ graphs. The following is an explanation of the graph that will be studied:

(a) The $Amal(F_n, xz, m)$ graph is the graph resulting from the operation amalgamation of the fan graph. A fan graph is denoted by $F_n$where $n \geq 3$ is a graph obtained from a path P_n by adding a point and joining all of n points, P_n [8]. Graph that resulted from operation amalgamation of fan graph that notated with $Amal(F_n, xz, m)$ with $n + 1$ is the number of points on the fan graph with m copy of the fan graph and xz as a fixed side.

(b) The $Amal(O_n, xz, m)$ graph is the graph resulting from the operation amalgamation of the fan graph. Octopus graph notated with O_n where $n \geq 3$ is formed from summation of fan graph and star graph. Graph that resulted from operation amalgamation of octopus graph is notated by $Amal(O_n, xz, m)$ with $2n + 1$ is the number of points on the fan graph with m copy of the octopus graph and xz as a fixed side.

In this research, there is a lemma used to make it easier to find rainbow connection number r. The lemma is shown as following.

Lemma 1[2] *For example G is a connected graph with size m, then $diam(G) \leq rc(G) \leq src(G) \leq m$, with $diam(G)$ is the diameter G and m is the number of sides of G.*

Lemma 2[3] *If graph G is nontrivially connected to size m, then*

(a) $src(G)$= 1 if and only if $G$is a complete graph,
(b) $rc(G)$= 2 if and only If $src(G) = 2$,
(c) $rc(G)$= m if and only if G is a tree graph.

2 Method Study

The method used in this study is the introduction pattern method and deduction axiomatic method. Deduction axiomatic is method based on valid deductive proof in logic mathematics. This method utilizes axioms, lemmas, and existing theorem from topic that being researched. Introduction pattern method (pattern recognition) is method used for knowing patterns, cardinality, and looking for rainbow connection number.

3 Results And Discussion

Theorem 1. *Rainbow connection number of graph $Amal(F_n, xz, m)$, for any integer $n \geq 3, m \geq 3$ is*

$$rc\big(Amal(F_n, xz, m)\big) = 3$$

Proof. $Amal(F_n, xz, m)$ graphs have sets of points $V\big(Amal(F_n, xz, m)\big) = \{x\} \cup \{z\} \cup \{x_{i,j}; 1 \leq i \leq m, 1 \leq j \leq n-1\}$ and sets of edges $E\big(Amal(F_n, xz, m)\big) = \{xz\} \cup \big\{xx_{i,1}; 1 \leq i \leq m\big\} \cup \big\{x_{i,j}x_{i,j+1}; 1 \leq i \leq m, 1 \leq j \leq n-2\big\} \cup \big\{zx_{i,j}; 1 \leq i \leq m, 1 \leq j \leq n-1\big\}$. The cardinality of the set of point is $\big|V\big(Amal(F_n, xz, m)\big)\big| = m(n-1) + 2$ and that of the set of edge is $\big|E\big(Amal(F_n, xz, m)\big)\big| = 2m(n-1) + 1$.

To prove $rc\big(Amal(F_n, xz, m)\big) = 3$ then it will be proven if $rc\big(Amal(F_n, xz, m)\big) \geq 3$ and $rc\big(Amal(F_n, xz, m)\big) \leq 3$. Next, we will prove the lower bound of $rc\big(Amal(F_n, xz, m)\big)$. Based on Lemma 1, we get $diam(Amal(F_n, xz, m)) = 2$ therefore $rc\big(Amal(F_n, xz, m)\big) \geq 2$. Then it will be proven that $rc\big(Amal(F_n, xz, m)\big) = 2$ is impossible. Assume $rc\big(Amal(F_n, xz, m)\big) = 2$, then $c: E\big(Amal(F_n, xz, m)\big) \to \{1,2\}$. For example, it is labeled side on $Amal(F_n, xz, m)$ graph as follows: $xz = 2, xx_{i,1} = 1, x_{i,j}x_{i,j+1} = 2$, $x_{i,j}z$ with $2; j \equiv 0(mod\ 2)$ and $1; j \equiv 1(mod\ 2)$. Then all of the $five$ points will be analyzed, namely $x, x_{1,2}, x_{2,1}, z$, and $x_{3,2}$. Next, it will be analyzed from point x to point $x_{1,2}$ and from point $x_{2,1}$ to point $x_{3,2}$ if there is *a* rainbow path. However, from point $x_{1,2}$ to the point $x_{3,2}$ there is no rainbow path because the trajectory formed own the same color. This matter show contradiction therefore $rc\big(Amal(F_n, xz, m)\big) \geq 3$.

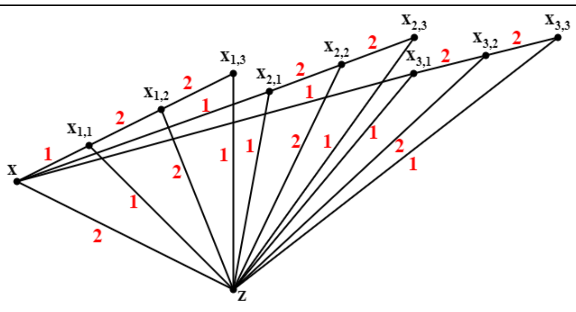
Next, we will prove the upper limit $rc\big(Amal(F_n, xz, m)\big)$ by defining the function $c: E\big(Amal(F_n, xz, m)\big) \rightarrow \{1,2,3\}$ as follows:

$$
\begin{aligned}
c(xz) &= 1\\
c(x_{i,j}z) &= \begin{cases} 1;\ j \equiv 0(mod\ 2), 1 \leq i \leq m, 1 \leq j \leq n-1 \\ 2;\ j \equiv 1(mod\ 2), 1 \leq i \leq m, 1 \leq j \leq n-1 \end{cases}\\
c(xx_{i,1}) &= 3;\\
c(x_{i,j}x_{i,j+1}) &= 3;\ 1 \leq i \leq m, 1 \leq j \leq n-2\\
c(y_{i,j}z) &= j + n(i-1);\ 1 \leq i \leq m, 1 \leq j \leq n-1
\end{aligned}
$$

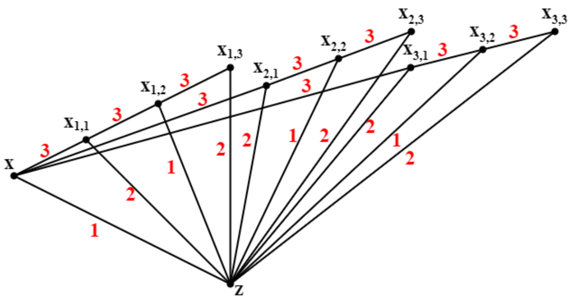
We can confirm from function c that the set of colors $A_1 = \{1,2,3\}$ such that $|A_1| = 3$. Because of that, $rc\big(Amal(F_n, xz, m)\big) \leq 3$. Based on the lower limit and limit from $rc\big(Amal(F_n, xz, m)\big)$, we get $3 \leq rc\big(Amal(F_n, xz, m)\big) \leq 3$. It is proven that $rc\big(Amal(F_n, xz, m)\big) = 3$. The rainbow path of the $Amal(F_n, xz, m))$ graph can be seen in Table 1. Illustration from coloring rainbow $Amal(F_n, xz, m))$ graph can be seen in Figure 1 with the same trajectory with no own rainbow path whereas different with trajectory on own rainbow path.

Table 1 Rainbow Path on $Amal(F_n, xz, m)$ Graph

Case	x	y	Rainbow Path	Condition
1	x	z	x, z	$-$
2	x	$x_{i,j}$	$x, z, x_{i,j}$	$j \equiv 1(mod\ 2), 1 \leq i \leq m, 1 \leq j \leq n-1$
3	x	$x_{i,j}$	$x, z, x_{i,j-1}, x_{i,j}$	$j \equiv 0(mod\ 2), 1 \leq i \leq m, 1 \leq j \leq n-1$
4	$x_{i,j}$	z	$x_{i,j}, z$	$1 \leq i \leq m, 1 \leq j \leq n-1$
5	$x_{i,j}$	$x_{k,l}$	$x_{i,j}, z, x_{k,l}$	$i,j \neq k,l;\ j \equiv 0(mod\ 2), l \equiv 1(mod\ 2)$ or $j \equiv 1(mod\ 2), l \equiv 0(mod\ 2)$
6	$x_{i,j}$	$x_{k,l}$	$x_{i,j}, z, x_{k,l-1}, x_{k,l}$	$i,j \neq k,l;\ j \equiv 1(mod\ 2), l \equiv 1(mod\ 2)$ or $j \equiv 0(mod\ 2), l \equiv 0(mod\ 2)$



(a) Trajectory The same



(b) Trajectory different

Figure 1 RC graph $Amal(F_n, xz, m)$

Theorem 2. *Rainbow connection number of $Amal(O_n, xz, m)$ graph, for any integer $n \geq 3$, $m \geq 2$ is*

$$rc\big(Amal(O_n, xz, m)\big) = mn$$

Proof. $Amal(O_n, xz, m)$ graph has a set of vertices $V\big(Amal(O_n, xz, m)\big) = \{x\} \cup \{z\} \cup \{x_{i,j}; 1 \leq i \leq m, 1 \leq j \leq n-1\} \cup \{y_{i,j}; 1 \leq i \leq n, 1 \leq j \leq m\}$ and a set of edges $E(Amal(O_n, xz, m)) = \{xz\} \cup \{xx_{i,1}; 1 \leq i \leq m\} \cup \{x_{i,j}x_{i,j+1}, 1 \leq i \leq m, 1 \leq j \leq n-2\} \cup \{zx_{i,j}; 1 \leq i \leq m,\ 1 \leq j \leq n-1\} \cup \{y_{i,j}z; 1 \leq i \leq m, 1 \leq j \leq n\}$. Cardinality from set of point is $|V(Amal(O_n, xz, m))| = m(2n-1) + 2$ and the set of edges is $|E(Amal(O_n, xz, m))| = m(3n-2) + 1$.

To prove $rc\big(Amal(O_n, xz, m)\big) = mn$ then it will be proven $rc\big(Amal(O_n, xz, m)\big) \geq mn$ and $rc\big(Amal(O_n, xz, m)\big) \leq mn$. Next, we will prove the lower bound of $rc\big(Amal(O_n, xz, m)\big)$. $Amal(O_n, xz, m)$ graph has a diameter of 2 and consists of an $Amal(F_n, xz, m)$ graph and a tree graph with *size mn*. Then it will be proven that $rc\big(Amal(O_n, xz, m)\big) = mn - 1$ is impossible. Assume $rc\big(Amal(O_n, xz, m)\big) = mn - 1$, then $c\colon E\big(Amal(O_n, xz, m)\big) \to \{1,2,\ldots, mn-1\}$. Then five points will be analyzed, namely $z, y_{1,1}, y_{2,1}, y_{3,1}$ and $y_{3,4}$ with $e\big(zy_{1,1}\big) = 1$, $e\big(zy_{2,1}\big) = 5$, $e\big(zy_{3,1}\big) = 9$, and $e\big(zy_{3,4}\big) = 1$. Next, it will be analyzed from $y_{1,1}y_{2,1}$ and $y_{1,1}y_{3,1}$ if there is a rainbow path. However, from the point from $y_{1,1}y_{3,4}$, there is no rainbow path because the formed trajectory own the same color. This matter shows contradiction therefore $rc\big(Amal(O_n, xz, m)\big) = mn - 1$ is impossible because the $Amal(O_n, xz, m)$ graph contains *a pendant* with *size mn*, then the labels of $zy_{i,j}$ with $1 \leq i \leq m$ and $1 \leq j \leq n$ must have different colors. Based on these assumptions and Lemma 2, then $rc\big(Amal(O_n, xz, m)\big) \geq mn$. Next, we will prove the upper limit by

$rc\big(Amal(O_n, xz, m)\big) \geq mn$ with defining the function $c: E\big(Amal(O_n, xz, m)\big) \rightarrow \big\{1,2,\dots, rc\big(Amal(O_n, xz, m)\big)\big\}$ as follows:

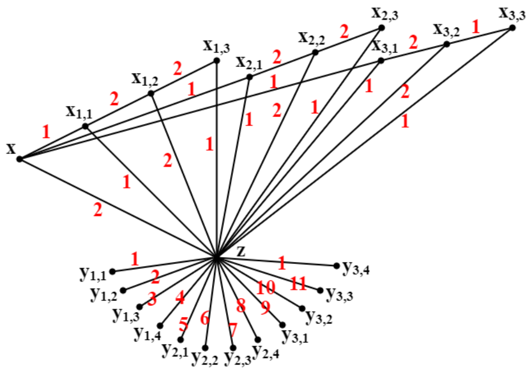
$$
\begin{aligned}
c(xz) &= 1 \\
c(x_{i,j}z) &= \begin{cases} 1;\ j \equiv 0(mod\ 2), 1 \leq i \leq m, 1 \leq j \leq n-1 \\ 2;\ j \equiv 1(mod\ 2), 1 \leq i \leq m, 1 \leq j \leq n-1 \end{cases} \\
c(xx_{i,1}) &= 3; \\
c(x_{i,j}x_{i,j+1}) &= 3;\ 1 \leq i \leq m, 1 \leq j \leq n-2 \\
c(y_{i,j}z) &= j + n(i-1);\ 1 \leq i \leq m, 1 \leq j \leq n-1
\end{aligned}
$$

This can be confirmed from function c that the set of colors $A_1 = \{1,2,3,\dots,mn\}$ such that $|A_1| = mn$. Therefore, the upper limit of $rc\big(Amal(O_m, xz, n)\big)$ is $rc\big(Amal(O_n, xz, m)\big) \leq mn$. Based on the lower limit and limit from $rc\big(Amal(O_3, xz, m)\big)$, $mn \leq rc\big(Amal(O_n, xz, m)\big) \leq mn$. It is proven that $rc\big(Amal(O_n, xz, m)\big) = mn$. The rainbow path of the $Amal(O_n, xz, m))$ graph can be seen in Table 2. Illustration of coloring rainbow $Amal(O_n, xz, m))$ graph can be seen in Figure 2 with the same trajectory with no own rainbow path whereas have different trajectory.

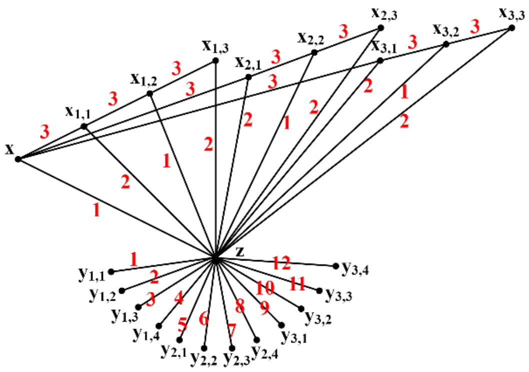
Table 2 Rainbow Path on $Amal(O_n, xz, m)$ Graph

Case	x	y	Rainbow Path	Condition
1	x	z	x, z	$-$
2	x	$x_{i,j}$	$x, z, x_{i,j}$	$1 \leq i \leq m, 1 \leq j \leq n-1$
3	x	$y_{i,j}$	$x, z, y_{i,j}$	$i,j \neq 1,1;\ 1 \leq i \leq m, 1 \leq j \leq n$
4	x	$y_{1,1}$	$x, x_{1,1}, z, y_{1,1}$	$-$
5	$x_{i,j}$	z	$x_{i,j}, z$	$1 \leq i \leq m, 1 \leq j \leq n-1$
6	$x_{i,j}$	$x_{k,l}$	$x_{i,j}, z, x_{k,l}$	$i,j \neq k,l;\ j \equiv 0(mod\ 2), l \equiv 1(mod\ 2)$ or $j \equiv 1(mod\ 2), l \equiv 0(mod\ 2)$
7	$x_{i,j}$	$x_{k,l}$	$x_{i,j}, z, x_{k,l-1}, x_{k,l}$	$i,j \neq k,l;\ j \equiv 1(mod\ 2), l \equiv 1(mod\ 2)$ or $j \equiv 0(mod\ 2), l \equiv 0(mod\ 2)$
8	$x_{i,j}$	$y_{k,l}$	$x_{i,j}, z, y_{k,l}$	$k,l \neq 1,1$ dan $k,l \neq 1,2; 1 \leq i \leq m,$ $1 \leq j \leq n-1,\ 1 \leq k \leq m,\ 1 \leq l \leq n$
9	$x_{i,j}$	$y_{1,1}$	$x_{i,j}, z, y_{1,1}$	$1 \leq i \leq m, 1 \leq j \leq n-1, j \equiv 1(mod\ 2)$
10	$x_{i,j}$	$y_{1,1}$	$x_{i,j}, x_{i,j-1}, z, y_{1,2}$	$1 \leq i \leq m, 1 \leq j \leq n-1, j \equiv 0(mod\ 2)$
11	$x_{i,j}$	$y_{1,2}$	$x_{i,j}, z, y_{1,2}$	$1 \leq i \leq m, 1 \leq j \leq n-1, j \equiv 0(mod\ 2)$

12	$x_{i,j}$	$y_{1,2}$	$x_{i,j}, x_{i,j-1}, z, y_{1,2}$	$1 \leq i \leq m, 1 \leq j \leq n-1, j \equiv 1(mod\ 2)$
13	$y_{i,j}$	z	$y_{i,j}, z$	$1 \leq i \leq m, 1 \leq j \leq n$
14	$y_{i,j}$	$y_{k,l}$	$y_{i,j}, z, y_{k,l}$	$i,j \neq k,l;\ 1 \leq i \leq m,\ 1 \leq k \leq m$ $1 \leq j \leq n,\ 1 \leq l \leq n$



(a) Trajectory the same



(b) Trajectory different

Figure 2 RC graph $Amal(O_n, xz, m)$

4 Conclusion

Based on the results and discussion, we obtain certain mark from rainbow connection number on $Amal(F_n, xz, m)$ graph and $Amal(O_n, xz, m)$ graph that is :

Theorem 1. *Rainbow connection number of* $Amal(F_n, xz, m)$ *graph, for any integer* $n \geq 3$, $m \geq 3$ *is* $rc(Amal(F_n, xz, m)) = 3$.

Theorem 2. *Rainbow connection number of* $Amal(O_n, xz, m)$ *graph, for any integer* $n \geq 3$, $m \geq 2$ *is* $rc(Amal(O_n, xz, m)) = mn$.

5 References

[1] Slamin, 2009, *Design Network: Approach Graph Theory*, Universitas Jember.

[2] Li, X. & Sun, Y., 2012, *Rainbow Connections of Graphs*, Springer Science & Business Media.

[3] Chartrand, G., Johns, G.L., McKeon, K.A. & Zhang, P., 2008, *Rainbow connections in graphs*, Mathematica Bohemica, **133**(1), 85-98.

[4] Kemnitz, A. & Schiermeyer, I., 2011, *Graphs with rainbow connection number two*, Discussiones Mathematicae Graph Theory, **31**(2), 313-320.

[5] Li, H., Li, X. & Liu, S., 2012, *Rainbow connection of graphs with diameter 2*, Discrete Mathematics, **312**(8), 1453–1457.

[6] Sy, S., Medika, G.H. & Yulianti, L., 2013, *The Rainbow Connection of Fan and Sun*, Applied Mathematical Sciences, **7**(64), 3155–3159.

[7] Fitriani, D. & Salman, A.N.M., 2016, *Rainbow connection number of amalgamation of some graphs*, AKCE International Journal of Graphs and Combinatorics, **13**(1), 90-99.

[8] Dafik, Susanto, F., Alfarisi, R., et al., 2021, *On rainbow antimagic coloring of graphs*, Advanced Mathematical Models & Applications, **6**(3), 278-291.