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The Szeged Index and Padmakar-Ivan Index on the Zero-Divisor Graph of a Commutative Ring

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Abstract. The zero-divisor graph of a commutative ring is a graph where the vertices represent the zero-divisors of the ring, and two distinct vertices are connected if their product equals zero. This study focuses on determining general formulas for the Szeged index and the Padmakar-Ivan index of the zero-divisor graph for specific commutative rings. The results show that for the first case of ring, the Szeged index is exactly half of the Padmakar-Ivan index. For the second xase, the Szeged index is consistently greater than the Padmakar-Ivan index. These findings enhance the understanding of how the algebraic structure of rings influences the topological properties of their associated graphs.

Keywords: *zero-divisor graph; Szeged index; Padmakar-Ivan index.*

1 Introduction

Chemical graph theory is a vibrant field of study that bridges the realms of mathematics and chemistry, enabling the analysis of molecular structures and properties through graph-theoretical approaches [1]. This interdisciplinary area has proven invaluable in understanding chemical reactivity, stability, and molecular interactions [2]. Within this framework, graph representations play a central role, translating complex algebraic and chemical structures into intuitive visual and computational models.

In group theory, notable examples include the coprime graph and non-coprime graph, which reveal intricate relationships among group elements based on divisibility criteria. Expanding on these algebraic structures, in ring theory, the zero-divisor graph and unit graph [3] encapsulate algebraic properties of commutative rings, providing insights into their structure and behavior.

To extract meaningful information from these representations, topological indices are widely used. Indices such as the Wiener index, Zagreb index, Gutman index, and Harmonic index quantify graph properties, offering a numerical lens through which structural and functional characteristics can be assessed [4]. These indices have found significant applications in fields ranging from chemistry—where they are used to predict

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molecular boiling points, stability, and reactivity—to computer science and algebraic studies.

This article narrows its focus to the zero-divisor graph of a commutative ring, a representation that uniquely captures the interplay between zero divisors within the ring structure. The primary aim is to derive general formulas for two important topological indices: the Szeged index, which is often used to evaluate molecular branching, and the Padmakar-Ivan (PI) index, known for its applications in characterizing molecular resonance energy and other chemical properties. By addressing these indices in the context of zero-divisor graphs, this study seeks to contribute both to algebraic graph theory and its potential applications in chemistry.

2 Literature Review

Topological indices, as a fundamental tool in graph theory, have been extensively studied for their ability to encode structural information about graphs. Widely used in chemical graph theory, these indices quantify graph properties that correlate with molecular characteristics. Among the most prominent are the Zagreb indices, which focus on vertex degrees to capture molecular branching; the Wiener index, which sums all pairwise distances to reflect molecular compactness [5]; and the Gutman index, which combines vertex degrees and distances for a more nuanced structural analysis [6]. Other critical indices include the Szeged index, which partitions edges into subgraphs based on distance metrics [7], and the Padmakar-Ivan (PI) index, a similar edge-based measure often applied in studying molecular bonds [8].

Recent research has extended the application of topological indices to zero-divisor graphs, providing new insights into their structural and algebraic interplay. Studies have analyzed indices such as the Zagreb, Wiener, and Gutman indices on zero-divisor graphs, uncovering relationships that reveal both graph-theoretic and algebraic properties [9]. Eccentricity-based indices have further enriched the field by offering perspectives on graph centrality and extremal properties.

While the Szeged and Padmakar-Ivan indices have been extensively studied in other graph contexts, such as chemical structures and nilpotent graphs, their application to zero-divisor graphs remains largely unexplored [10]. Existing work has focused on these indices in various graph classes, highlighting their versatility but leaving a gap in their application to zero-divisor graphs of commutative rings.

This study addresses this research gap by deriving general formulas for the Szeged and Padmakar-Ivan indices for zero-divisor graphs of commutative rings modulo integers. By doing so, it seeks to enhance the understanding of how algebraic properties of rings influence the structural descriptors of their associated graphs. These findings aim to bridge the gap between algebraic graph theory and chemical graph theory, paving the way for broader applications in both fields.

3 Result

Based on the literature review conducted, several lemmas related to the structure of zero-divisor graphs and theorems concerning the Szeged index and Padmakar-Ivan index of zero-divisor graphs from commutative rings were obtained as follows:

Lemma 3.1 Order and sized of the graph $\Gamma(\mathbb{Z}_{p^2})$ with p a odd prime number respectively are $p-1$ and $\frac{1}{2}(p-1)(p-2)$.

Proof. We have $\mathbb{Z}_{p^2} = \{\bar{0}, \bar{1}, \bar{2}, \ldots, \overline{p^2-1}\}$ with p a odd prime number. Based on the definition of zero-divisor, then $Z(\mathbb{Z}_{p^2}) = \{\bar{p}, \overline{2p}, \overline{3p}, \ldots, \overline{p(p-1)}\}$. Hence $|Z(\mathbb{Z}_{p^2})| = p-1$. Then we have $\Gamma(\mathbb{Z}_{p^2})$ is $p-1$. And based on the definition of zero-divisor graph, the edge $\overline{pi}$ with $i = 1, 2, \ldots, p-1$ are neighbor and graph $\Gamma(\mathbb{Z}_{p^2})$ It can be illustrated as shown in Figure 1 below.

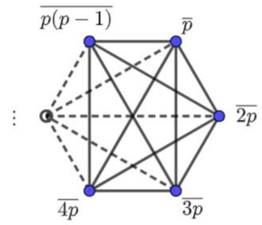

Figure 1 The graph of $\Gamma(\mathbb{Z}_{p^2})$

Based on Figure 1, the zero divisor graph of a commutative ring $\Gamma(\mathbb{Z}_{p^2})$ is complete graph with order $p-1$ hence $\Gamma(\mathbb{Z}_{p^2}) \cong K_{p-1}$. Because every two vertices are adjacent to each other, hence there are $C_2^{p-1} = \frac{1}{2}(p-1)(p-2)$ edges. Therefore, the size of the graph $\Gamma(\mathbb{Z}_{p^2})$ is $\|\Gamma(\mathbb{Z}_{p^2})\| = \frac{1}{2}(p-1)(p-2)$. ■

Theorem 3.1 Let $\mathbb{Z}_{p^2}$ be a commutative ring where p is odd prime number. The Szeged index of the zero divisor graph of the commutative ring $\Gamma(\mathbb{Z}_{p^2})$ is $Sz\left(\Gamma(\mathbb{Z}_{p^2})\right) = \frac{1}{2}(p-1)(p-2)$.

Proof. Based on Lemma 3.1, the size of the graph $\|\Gamma(\mathbb{Z}_{p^2})\| = \frac{1}{2}(p-1)(p-2)$ with $\Gamma(\mathbb{Z}_{p^2}) \cong K_{p-1}$. Since the resulting graph is a complete graph, every two vertices are adjacent. Therefore, for $e = uv$ it follows that $n_u(e|\mathbb{Z}_{p^2}) = |\{u\}| = 1$ and $n_v(e|\mathbb{Z}_{p^2}) = |\{v\}| = 1$ for every $u, v \in Z(\mathbb{Z}_{p^2})$. Considering the formula for the Szeged index, which is defined as the product of the cardinalities of the smallest distances from two vertices for each edge in the graph $\Gamma(\mathbb{Z}_{p^2})$, the Szeged index of the zero divisor graph of the commutative ring $\mathbb{Z}_{p^2}$ is

$$\begin{aligned} Sz\left(\Gamma(\mathbb{Z}_{p^2})\right) &= \sum_{e\in E\left(\Gamma(\mathbb{Z}_{p^2})\right)} n_u(e|\mathbb{Z}_{p^2}) \cdot n_v(e|\mathbb{Z}_{p^2}) \\ &= \left(\frac{1}{2}(p-1)(p-2)\right) \cdot (1 \cdot 1) \\ Sz\left(\Gamma(\mathbb{Z}_{p^2})\right) &= \frac{1}{2}(p-1)(p-2). \end{aligned}$$

Thus, it is proven that the Szeged index of the zero divisor graph of the commutative ring $\Gamma(\mathbb{Z}_{p^2})$ where p is odd prime is $Sz\left(\Gamma(\mathbb{Z}_{p^2})\right) = \frac{1}{2}(p-1)(p-2)$. ∎

Theorem 3.2 Let $\mathbb{Z}_{p^2}$ be a commutative ring where p is odd prime. The Padmakar-Ivan index of the zero divisor graph of the commutative ring $\mathbb{Z}_{p^2}$ is $PI\left(\Gamma(\mathbb{Z}_{p^2})\right) = (p-1)(p-2)$.

Proof. In the same manner as Theorem 3.1, but defined as the sum of the cardinalities of the smallest distances between two vertices for each edge in the graph $\Gamma(\mathbb{Z}_{p^2})$, the Padmakar-Ivan index of the zero divisor graph of the commutative ring $\mathbb{Z}_{p^2}$ is

$$\begin{aligned} PI\left(\Gamma(\mathbb{Z}_{p^2})\right) &= \sum_{e\in E\left(\Gamma(\mathbb{Z}_{p^2})\right)} n_u(e|\mathbb{Z}_{p^2}) + n_v(e|\mathbb{Z}_{p^2}) \\ &= \left(\frac{1}{2}(p-1)(p-2)\right) \cdot (1 + 1) \\ PI\left(\Gamma(\mathbb{Z}_{p^2})\right) &= (p-1)(p-2). \end{aligned}$$

Thus, it is proven that the Padmakar-Ivan index of the zero divisor graph of the commutative ring $\mathbb{Z}_{p^2}$ where p is odd prime is $PI\left(\Gamma(\mathbb{Z}_{p^2})\right) = (p-1)(p-2)$. ∎

Lemma 3.2 The order and size of the graph $\Gamma(\mathbb{Z}_{3q})$, where q is prime, $q \geq 5$, are $q+1$ and $2q-2$ respectively.

Proof. et the elements of the commutative ring $\mathbb{Z}_{3q}$ be $\mathbb{Z}_{3q} = \{\overline{0}, \overline{1}, \overline{2}, \dots, \overline{3q-1}\}$ where q is prime, $q \geq 5$. Based on the definition of zero divisors, $\mathbb{Z}_{3q}$ has the set of zero divisors $Z(\mathbb{Z}_{3q}) = \{\overline{3}, \overline{6}, \overline{9}, \dots, \overline{3(q-1)}, \overline{q}, \overline{2q}\}$. Based on the definition of a zero divisor graph, the vertex $\overline{3\imath}$ where $i = 1, 2, \dots, q-1$ will only be adjacent to the vertices $\overline{q\jmath}$ where $j = 1, 2$. Therefore, the graph $\Gamma(\mathbb{Z}_{3q})$ can be illustrated as shown in Figure 2 below.

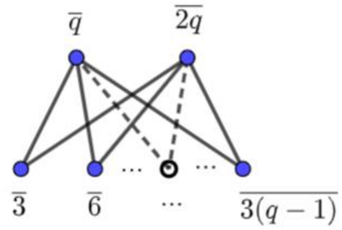

Figure 2 The graph of $\Gamma(\mathbb{Z}_{3q})$

Based on Figure 2, the zero divisor graph of the commutative ring $\mathbb{Z}_{3q}$ is a complete bipartite graph, where the number of vertices in the first partition is 2, and in the second partition is $q-1$. Thus, the order of the graph $\Gamma(\mathbb{Z}_{3q})$ is $\left|Z\left(\Gamma(\mathbb{Z}_{3q})\right)\right| = 2 + q - 1 = q + 1$ and $\Gamma(\mathbb{Z}_{3q}) \cong K_{2,q-1}$. Since every two vertices in different partitions are adjacent, we have $\|\Gamma(\mathbb{Z}_{3q})\| = 2 \cdot (q-1)$. Therefore, the size of the graph $\Gamma(\mathbb{Z}_{3q})$ is $2q-2$. ■

Lemma 3.3 Let $\mathbb{Z}_{3q}$ be a commutative ring where q prime number and $q \geq 5$, then the Szeged index of the zero divisor graph of the commutative ring $\mathbb{Z}_{3q}$ is $Sz\left(\Gamma(\mathbb{Z}_{3q})\right) = 2^2 \cdot (q-1)^2$.

Proof. Based on Lemma 3.1, the size of the graph $\|\Gamma(\mathbb{Z}_{3q})\| = 2(q-1)$ with $\Gamma(\mathbb{Z}_{3q}) \cong K_{2,q-1}$. Since the resulting graph is a complete bipartite graph, every two vertices in different partitions are adjacent. If uv is an edge, then it follows that $n_u(e|\mathbb{Z}_{3q}) = 2$ and $n_v(e|\mathbb{Z}_{3q}) = q-1$ or $n_u(e|\mathbb{Z}_{3q}) = q-1$ and $n_v(e|\mathbb{Z}_{3q}) = 2$. Considering the formula for the Szeged index, which is defined as the product of the cardinalities of the smallest distances between two vertices for each edge in the graph $\Gamma(\mathbb{Z}_{3q})$, the Szeged index of the zero divisor graph of the commutative ring $\mathbb{Z}_{3q}$ is

$$Sz\left(\Gamma(\mathbb{Z}_{3q})\right) = \sum_{e \in E\left(\Gamma(\mathbb{Z}_{3q})\right)} n_u(e|\mathbb{Z}_{3q}) \cdot n_v(e|\mathbb{Z}_{3q})$$

$$= 2(q-1)\cdot 2(q-1)$$
$$Sz\left(\Gamma(\mathbb{Z}_{3q})\right) = 2^2\cdot(q-1)^2.$$

Thus, it is proven that the Szeged index of the zero divisor graph of the commutative ring $\Gamma(\mathbb{Z}_{3q})$ where q is prime and $q \geq 5$ is $Sz\left(\Gamma(\mathbb{Z}_{3q})\right) = 2^2(q-1)^2$. ■

Lemma 3.4 Let $\mathbb{Z}_{3q}$ be a commutative ring where q is prime and $q \geq 5$, then the Padmakar-Ivan index of the zero divisor graph of the commutative ring $\mathbb{Z}_{3q}$ is $PI\left(\Gamma(\mathbb{Z}_{3q})\right) = 2\cdot(q^2-1)$.

Proof. In the same manner as Lemma 3.3, but defined as the sum of the cardinalities of the smallest distances between two vertices for each edge in the graph $\Gamma(\mathbb{Z}_{3q})$, the Padmakar-Ivan index of the zero divisor graph of the commutative ring $\mathbb{Z}_{3q}$ is

$$PI\left(\Gamma(\mathbb{Z}_{3q})\right) = \sum_{e\in E\left(\Gamma(\mathbb{Z}_{3q})\right)} n_u(e|\mathbb{Z}_{3q}) + n_v(e|\mathbb{Z}_{3q})$$
$$= \left(2+(q-1)\right)\cdot 2(q-1)$$
$$= (q+1)\cdot 2(q-1)$$
$$PI\left(\Gamma(\mathbb{Z}_{3q})\right) = 2\cdot(q^2-1).$$

Thus, it is proven that the Padmakar-Ivan index of the zero divisor graph of the commutative ring $\mathbb{Z}_{3q}$ with q is prime number $q \geq 5$ is $PI\left(\Gamma(\mathbb{Z}_{3q})\right) = 2\cdot(q^2-1)$. ■

Lemma 3.5 The order and size of the graph $\Gamma(\mathbb{Z}_{5q})$ with q is prime number $q \geq 7$ is $q+3$ and $4(q-1)$ respectively.

Proof. Let the elements of the commutative ring $\mathbb{Z}_{5q}$ is $\mathbb{Z}_{5q} = \{\overline{0}, \overline{1}, \overline{2}, \dots, \overline{5q-1}\}$ where q is prime number and $q \geq 7$. Based on the definition of zero divisors, $\mathbb{Z}_{5q}$ has the set of zero divisors $Z(\mathbb{Z}_{5q}) = \{\overline{5}, \overline{10}, \overline{15}, \dots, \overline{5(q-1)}, \overline{q}, \overline{2q}, \overline{3q}, \overline{4q}\}$. Based on the definition of a zero divisor graph, the vertex $\overline{5\imath}$ where $i = 1, 2, \dots, q-1$ will only be adjacent to the vertices $\overline{q\jmath}$ where $j = 1, 2, 3, 4$. Therefore, the graph $\Gamma(\mathbb{Z}_{5q})$ can be illustrated as shown in Figure 3.

Based on the diagram of the graph $\Gamma(\mathbb{Z}_{5q})$, the zero divisor graph of the commutative ring $\mathbb{Z}_{5q}$ is a complete bipartite graph, where the number of vertices in the first partition is 4 and in the second partition is $q-1$. Therefore, the order of the graph $\Gamma(\mathbb{Z}_{5q})$ is $\left|\left(\Gamma(\mathbb{Z}_{5q})\right)\right| = 4+q-1 = q+3$ and $\Gamma(\mathbb{Z}_{5q}) \cong K_{4,q-1}$. Since every two vertices in different partitions are adjacent, it follows that $\|\Gamma(\mathbb{Z}_{5q})\| = 4 \cdot (q-1)$. Therefore, the size of the graph $\Gamma(\mathbb{Z}_{5q})$ is $4(q-1)$. ■

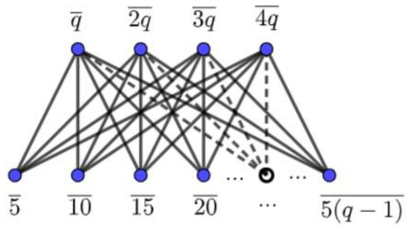

Figure 3 Graph $\Gamma(\mathbb{Z}_{5q})$

Lemma 3.6 Let $\mathbb{Z}_{5q}$ be a commutative ring where q is prime number $q \geq 5$, then the Szeged index of the zero divisor graph of the commutative ring $\mathbb{Z}_{5q}$ is $Sz\left(\Gamma(\mathbb{Z}_{5q})\right) = 4^2 \cdot (q-1)^2$.

Proof. Based on Lemma 3.5, the size of the graph is $\|\Gamma(\mathbb{Z}_{5q})\| = 4(q-1)$ with $\Gamma(\mathbb{Z}_{5q}) \cong K_{4,q-1}$. Since the graph formed is a complete bipartite graph, every two vertices in different partitions are adjacent. If uv and edge, it follow that $n_u(e|\mathbb{Z}_{5q}) = 4$ and $n_v(e|\mathbb{Z}_{5q}) = q-1$ or $n_u(e|\mathbb{Z}_{5q}) = q-1$ and $n_v(e|\mathbb{Z}_{5q}) = 4$. Considering the Szeged index formula, defined as the product of the cardinalities of the smallest distances between two vertices on each edge in the graph $\Gamma(\mathbb{Z}_{5q})$, the Szeged index of the zero divisor graph of the commutative ring $\mathbb{Z}_{5q}$ is

$$\begin{aligned} Sz\left(\Gamma(\mathbb{Z}_{5q})\right) &= \sum_{e \in E\left(\Gamma(\mathbb{Z}_{5q})\right)} n_u(e|\mathbb{Z}_{5q}) \cdot n_v(e|\mathbb{Z}_{5q}) \\ &= 4(q-1) \cdot 4(q-1) \\ Sz\left(\Gamma(\mathbb{Z}_{5q})\right) &= 4^2 \cdot (q-1)^2. \end{aligned}$$

Thus, it is proven that the Szeged index of the zero divisor graph of the commutative ring $\Gamma(\mathbb{Z}_{5q})$ with q is prime number and $q \geq 7$ is $Sz\left(\Gamma(\mathbb{Z}_{5q})\right) = 4^2 \cdot (q-1)^2$. ■

Lemma 3.7 Let $\mathbb{Z}_{5q}$ be a commutative ring with q a prime number and $q \geq 7$, then the Padmakar-Ivan index of the zero divisor graph of the commutative ring $\mathbb{Z}_{5q}$ is $PI\left(\Gamma(\mathbb{Z}_{5q})\right) = 4 \cdot (q^2 + 2q - 3)$.

Proof. In the same way as Lemma 3.6, but defined as the sum of the cardinalities of the shortest distances between two vertices on each edge of the graph $\Gamma(\mathbb{Z}_{5q})$, the Padmakar-Ivan index of the zero divisor graph of the commutative ring $\mathbb{Z}_{5q}$ is

$$\begin{aligned} PI\left(\Gamma(\mathbb{Z}_{5q})\right) &= \sum_{e \in E\left(\Gamma(\mathbb{Z}_{5q})\right)} n_u(e|\mathbb{Z}_{5q}) + n_v(e|\mathbb{Z}_{5q}) \\ &= \left(4 + (q-1)\right) \cdot 4(q-1) \\ &= (q+3) \cdot 4(q-1) \\ PI\left(\Gamma(\mathbb{Z}_{5q})\right) &= 4 \cdot (q^2 + 2q - 3). \end{aligned}$$

Thus, it is proven that the Padmakar-Ivan index of the zero divisor graph of the commutative ring $\mathbb{Z}_{5q}$ with q a prime number and $q \geq 7$ is $PI\left(\Gamma(\mathbb{Z}_{5q})\right) = 4 \cdot (q^2 + 2q - 3)$. ■

Lemma 3.8 The order and size of the graph $\Gamma(\mathbb{Z}_{7q})$ with q a prime number and $q \geq 11$ are $q + 5$ and $6(q-1)$, respectively.

Proof. Let the elements of the commutative ring $\mathbb{Z}_{7q}$ is $\mathbb{Z}_{7q} = \{\overline{0}, \overline{1}, \overline{2}, \dots, \overline{7q-1}\}$ where q a prime number and $q \geq 11$. Based on the definition of zero divisors $\mathbb{Z}_{7q}$ has the set of zero divisors $Z(\mathbb{Z}_{7q}) = \{\overline{7}, \overline{14}, \overline{21}, \dots, \overline{7(q-1)}, \overline{q}, \overline{2q}, \dots, \overline{6q}\}$. According to the definition of the zero divisor graph, the vertex $\overline{7i}$ where $i = 1, 2, \dots, q-1$ will only be adjacent to the vertex $\overline{qj}$ where $j = 1, 2, \dots, 6$ so the graph $\Gamma(\mathbb{Z}_{7q})$ can be illustrated as shown in Figure 4 below.

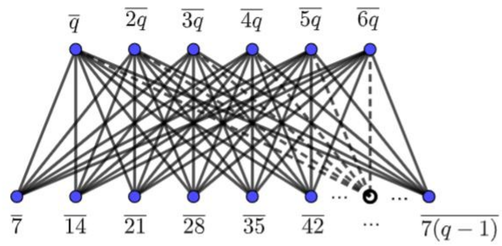

Figure 4 Graf $\Gamma(\mathbb{Z}_{7q})$

Based on Figure 4, the zero divisor graph of the commutative ring $\Gamma(\mathbb{Z}_{7q})$ is a complete bipartite graph where the number of vertices in the first partition is 6 and in the second partition is $q-1$. Therefore, the order of the graph $\Gamma(\mathbb{Z}_{7q})$ is $|\Gamma(\mathbb{Z}_{7q})| = 6 + q - 1 = q + 5$ and $\Gamma(\mathbb{Z}_{7q}) \cong K_{6,q-1}$. Since every two vertices in different partitions are adjacent, we have $\|\Gamma(\mathbb{Z}_{7q})\| = 6 \cdot (q-1)$. Therefore, the size of the graph $\Gamma(\mathbb{Z}_{7q})$ is $6(q-1)$. ■

Lemma 3.9 Let $\mathbb{Z}_{7q}$ be a commutative ring with q a prime number and $q \geq 11$, then the Szeged index of the zero divisor graph of the commutative ring $\mathbb{Z}_{7q}$ is $Sz\left(\Gamma(\mathbb{Z}_{7q})\right) = 6^2 \cdot (q-1)^2$.

Proof. Based on Lemma 3.8, the size of the graph $\|\Gamma(\mathbb{Z}_{7q})\| = 6(q-1)$ with $\Gamma(\mathbb{Z}_{5q}) \cong K_{6,q-1}$. Since the graph formed is a complete bipartite graph, every two vertices from different partitions are adjacent. Therefore, we have $n_u(e|\mathbb{Z}_{7q}) = 6$ and $n_v(e|\mathbb{Z}_{7q}) = q-1$ or $n_u(e|\mathbb{Z}_{7q}) = q-1$ and $n_v(e|\mathbb{Z}_{7q}) = 6$. Considering the Szeged index formula, which is defined as the product of the smallest distance cardinality between two vertices for every edge in the graph $\Gamma(\mathbb{Z}_{7q})$, the Szeged index of the zero divisor graph of the commutative ring $\mathbb{Z}_{7q}$ is

$$\begin{aligned}
Sz\left(\Gamma(\mathbb{Z}_{7q})\right) &= \sum_{e \in E\left(\Gamma(\mathbb{Z}_{7q})\right)} n_u(e|\mathbb{Z}_{7q}) \cdot n_v(e|\mathbb{Z}_{7q}) \\
&= 6(q-1) \cdot 6(q-1) \\
Sz\left(\Gamma(\mathbb{Z}_{7q})\right) &= 6^2 \cdot (q-1)^2.
\end{aligned}$$

Thus, it is proven that the Szeged index of the zero divisor graph of the commutative ring $\Gamma(\mathbb{Z}_{7q})$ with q a prime number and $q \geq 11$ is $Sz\left(\Gamma(\mathbb{Z}_{7q})\right) = 6^2 \cdot (q-1)^2$. ■

Lemma 3.10 Let $\mathbb{Z}_{7q}$ be a commutative ring with q a prime number and $q \geq 11$, then the Padmakar-Ivan index of the zero divisor graph of the commutative ring $\mathbb{Z}_{7q}$ is $PI\left(\Gamma(\mathbb{Z}_{7q})\right) = 6 \cdot (q^2 + 4q - 5)$.

Proof. In the same way as Lemma 3.9, but defined as the sum of the smallest distance cardinalities between two vertices on each edge of the graph $\Gamma(\mathbb{Z}_{7q})$, the Padmakar-Ivan index of the zero divisor graph of the commutative ring $\mathbb{Z}_{7q}$ is

$$\begin{aligned}
PI\left(\Gamma(\mathbb{Z}_{7q})\right) &= \sum_{e \in E\left(\Gamma(\mathbb{Z}_{7q})\right)} n_u(e|\mathbb{Z}_{7q}) + n_v(e|\mathbb{Z}_{7q}) \\
&= \left(6 + (q-1)\right) \cdot 6(q-1) \\
&= (q+5) \cdot 6(q-1) \\
PI\left(\Gamma(\mathbb{Z}_{7q})\right) &= 6 \cdot (q^2 + 4q - 5).
\end{aligned}$$

Thus, it is proven that the Padmakar-Ivan index of the zero divisor graph of the commutative ring $\mathbb{Z}_{7q}$ with q a prime and $q \geq 11$ is $PI\left(\Gamma(\mathbb{Z}_{7q})\right) = 6 \cdot (q^2 + 4q - 5)$. ■

Lemma 3.11 The order and size of the graph $\Gamma(\mathbb{Z}_{pq})$ with p and q prime, $p \geq 3$, and $p < q$ are respectively $p + q - 2$ dan $(p-1)(q-1)$.

Proof. Let the elements of the commutative ring $\mathbb{Z}_{pq}$ be $\mathbb{Z}_{pq} = \{\bar{0}, \bar{1}, \bar{2}, \dots, \overline{pq-1}\}$ for p and q prime and $p < q$. Based on the definition of zero divisors, the set of zero divisors of $Z(\mathbb{Z}_{pq}) = \{\bar{p}, \overline{2p}, \overline{3p}, \dots, \overline{q(p-1)}, \bar{q}, \overline{2q}, \overline{3q}, \dots, \overline{p(q-1)}\}$. Based on the definition of a zero divisor graph, a vertex $\overline{pi}$ where $i = 1, 2, \dots, q-1$ will only be adjacent to vertices $\overline{qj}$ where $j = 1, 2, \dots, p-1$ so the grap $\Gamma(\mathbb{Z}_{pq})$ can be represented as shown in Figure 5 below.

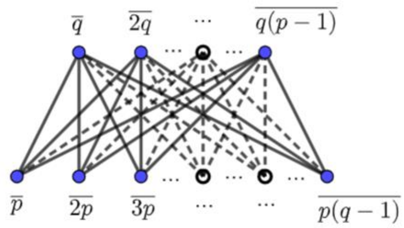

Figure 5 Graph $\Gamma(\mathbb{Z}_{pq})$

Based on Figure 5, the zero divisor graph of the commutative ring $\Gamma(\mathbb{Z}_{pq})$ is a complete bipartite graph, where the number of vertices in the first partition is $p-1$ and in the second partition is $q-1$. Therefore, the order of the graph $\Gamma(\mathbb{Z}_{pq})$ is $|\Gamma(\mathbb{Z}_{pq})| = p - 1 + q - 1 = p + q - 2$ and $\Gamma(\mathbb{Z}_{pq}) \cong K_{p-1,q-1}$. Since every two vertices in different partitions are adjacent, we have: $\|\Gamma(\mathbb{Z}_{pq})\| = (p-1) \cdot (q-1)$. Therefore, the size of the graph $\Gamma(\mathbb{Z}_{pq})$ is $(p-1)(q-1)$. ■

Theorem 3.3 Let $\mathbb{Z}_{pq}$ be a commutative ring for prime numbers p and q, $p \geq 3$, and $p < q$. The Szeged index of the zero divisor graph of the commutative ring $\mathbb{Z}_{pq}$ adalaish $Sz\left(\Gamma(\mathbb{Z}_{pq})\right) = (p-1)^2 \cdot (q-1)^2$.

Proof. Based on Lemma 3.11, the size of the graph is given by $\|\Gamma(\mathbb{Z}_{pq})\| = (p-1)(q-1)$ with $\Gamma(\mathbb{Z}_{pq}) \cong K_{p-1,q-1}$. Since the graph formed is a complete bipartite graph, every two vertices in different partitions are adjacent. Therefore, we have $n_u(e|\mathbb{Z}_{pq}) = p - 1$ and $n_v(e|\mathbb{Z}_{pq}) = q - 1$ or $n_u(e|\mathbb{Z}_{pq}) = q - 1$ and $n_v(e|\mathbb{Z}_{pq}) = p - 1$. Considering the Szeged index formula, which is defined as the product of the cardinalities of the shortest distances between two vertices on every edge in the graph $\Gamma(\mathbb{Z}_{pq})$, the Szeged index for the zero divisor graph of the commutative ring $\mathbb{Z}_{pq}$ is

$$\begin{aligned} Sz\left(\Gamma(\mathbb{Z}_{pq})\right) &= \sum_{e\in E\left(\Gamma(\mathbb{Z}_{pq})\right)} n_u(e|\mathbb{Z}_{pq}) \cdot n_v(e|\mathbb{Z}_{pq}) \\ &= (p-1)(q-1) \cdot (p-1)(q-1) \\ Sz\left(\Gamma(\mathbb{Z}_{pq})\right) &= (p-1)^2 \cdot (q-1)^2. \end{aligned}$$

Therefore, it is proven that the Szeged index for the zero divisor graph of the commutative ring $\Gamma(\mathbb{Z}_{pq})$ with p and q prime, $p \geq 3$, and $p < q$ adalah $Sz\left(\Gamma(\mathbb{Z}_{pq})\right) = (p-1)^2 \cdot (q-1)^2$. ■

Theorem 3.4 Let $\mathbb{Z}_{pq}$ be a commutative ring for p and q prime, $p \geq 3$, and $p < q$. The Padmakar-Ivan index for the zero divisor graph of the commutative ring $\mathbb{Z}_{pq}$ is $PI\left(\Gamma(\mathbb{Z}_{pq})\right) = (p-1)(q-1) \cdot (p+q-2)$.

Proof. In the same way as Theorem 3.3, but defined as the sum of the cardinalities of the shortest distances between two vertices on each edge of the graph $\Gamma(\mathbb{Z}_{pq})$, the Padmakar-Ivan index for the zero divisor graph of the commutative ring $\mathbb{Z}_{pq}$ is

$$\begin{aligned} PI\left(\Gamma(\mathbb{Z}_{pq})\right) &= \sum_{e\in E\left(\Gamma(\mathbb{Z}_{pq})\right)} n_u(e|\mathbb{Z}_{pq}) + (e|\mathbb{Z}_{pq}) \\ &= (p-1)(q-1) \cdot (p-1) + (q-1) \\ PI\left(\Gamma(\mathbb{Z}_{pq})\right) &= (p-1)(q-1) \cdot (p+q-2). \end{aligned}$$

Thus, it is proven that the Padmakar-Ivan index of the zero divisor graph of the commutative ring $\mathbb{Z}_{pq}$ with p and q prime, $p \geq 3$,and $p < q$ is $PI\left(\Gamma(\mathbb{Z}_{pq})\right) = (p-1)(q-1) \cdot (p+q-2)$. ■

4 Conclusions

This study explored the Szeged and Padmakar-Ivan indices for the zero-divisor graph of specific commutative rings. By focusing on rings formed by integers modulo, we derived general formulas for these indices based on the structural properties of their graphs. The findings reveal a clear relationship between the algebraic properties of the rings and the topological characteristics of their associated zero-divisor graphs. For the first case, the Szeged index is precisely half of the Padmakar-Ivan index, reflecting the symmetrical and uniform structure of the graph. In contrast, for the the second case, the Szeged index is consistently higher, demonstrating the influence of increased complexity in the ring's algebraic structure. These results not only contribute to algebraic graph theory but also highlight the broader applicability of topological indices. Future research may explore extensions of these indices to other graph classes or algebraic structures.

5 References

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