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Existence and Uniqueness of Homogeneous Linear Equation Solutions in Supertropical Algebra

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Abstract. This research investigates the existence and uniqueness of solutions to homogeneous linear equations in supertropical algebra. We analyze the structure of supertropical matrices to identify the conditions in which nontrivial solutions exist for the system of equations $A\otimes x \vDash \varepsilon$, where A is a matrix over a supertropical semiring and x is a vector. By applying determinant-based criteria, we demonstrate how tropical and supertropical values influence the solution space. The research applies theorems that determine the presence of trivial and nontrivial solutions and uses examples to illustrate practical methods for solving homogeneous matrix systems. This highlights the distinct characteristics of supertropical algebra compared to classical linear algebra. Our findings provide a deeper insight into solution behaviors in supertropical systems, paving the way for further research in tropical mathematics.

Keywords: *homogeneous linear equations; matrix system; max-plus; supertropical algebra; tropical.*

1 Introduction

Tropical algebra is a branch of mathematics that developed in the 1980s and was initially introduced by Imre Simon [1]. As an idempotent semiring and semifield, tropical algebra provides a novel approach to solving mathematical problems with a structure distinct from classical algebra [2]. One example of tropical algebra is the max-plus algebra, which replaces addition with the maximum operation and multiplication with addition. It offers a unique structure that differs from classical algebra, where the maximum operation replaces addition, and multiplication is replaced by addition, as seen in max-plus algebra.

As tropical algebra evolved, supertropical algebra emerged as a further extension, addressing some of the limitations of max-plus algebra. Supertropical algebra offers a more general structure, allowing for more complex polynomial analysis and manipulation [3]. Research in supertropical algebra covers various topics, including polynomial factorization, matrix theory, and valuation theory, with many studies aimed at understanding the structure and applications of this algebra [4-7].

In supertropical algebra, studying linear systems of equations is essential because of their relevance and practical applications [8]. Homogeneous linear systems, in particular, are notable for always having a trivial solution, but their other solutions can vary based on the properties of the system. Research on solutions to these homogeneous linear systems in supertropical algebra is essential as it helps expand current theories and offers new insights into how solutions can be characterized within this more complex algebraic framework.

This research aims to address a gap in the current literature by analyzing solutions to homogeneous linear systems within supertropical algebra in depth. By applying existing theories of matrices and characteristic polynomials, this study seeks to enhance our understanding and application of supertropical algebra in solving linear systems. The findings could also have practical applications in areas like optimization and control theory, where linear systems are commonly used.

2 Method

The theoretical basis of this research is rooted in supertropical algebra, with a focus on exploring the existence and uniqueness of solutions for homogeneous linear systems represented by $A \otimes x \vDash \varepsilon$.

2.1 Maxplus Algebra

Definition 2.1.1. [7] Let $R_\epsilon = R \cup \{\varepsilon\}$ be the set that includes all real numbers along with $\varepsilon = -\infty$. We define two operations on R_ϵ as follows: for any $a, b \in R$,

$$a \oplus b = \max\{a, b\} \tag{1}$$

$$a \otimes b = a + \mathrm{b}$$

Then $(R, \oplus, \otimes)$ is referred to as max-plus algebra, where $\varepsilon = -\infty$ is the neutral element and $e = 0$ is the identity element for the operation $\otimes$.

Furthermore $(R, \oplus, \otimes)$ is a semifield, it is a commutative semiring where, for every $a \in R$, there exists $-a$ such that $a \otimes - a = - a \otimes a = 0$. Maxplus algebra is a particular type of tropical algebra and is denoted as R_{max}.

The operations $\oplus$ and $\otimes$ in R_{max} can be extended to matrices in $R_{max}^{m\times n}$, where

$$R_{max}^{m\times n} = \{A = (A_{ij}) \,|\, A_{ij} \in R_{max}, i = 1,2,\ldots,m \text{ } and \text{ } j = 1,2,\ldots,n\}.$$

Definition 2.1.2. [9] For matrices $A, B \in R_{max}^{m\times n}$ and $\alpha \in R$ the operations are defined as follows:

$$(A \oplus B)_{ij} = a_{ij} + b_{ij} \text{ and } \alpha \otimes (B)_{ij} = \alpha \otimes b_{ij} \tag{2}$$

Definition 2.1.3. [7] For matrices $A \in R_{max}^{n\times p}$ and $B \in R_{max}^{p\times m}$ the matrix product is defined by:

$$(A \otimes B)_{ij} = \oplus_{k=1}^{n} a_{ik} \otimes b_{kj} \quad (3)$$

We define $\mathbb{R}_{max}^{n} = \{[x_1, x_2, \dots, x_n\ ]^T | x_i \in \mathbb{R}_{max},\ i = 1,2,\dots,n\ \}$. This set can be seen as $\mathbb{R}_{max}^{n\times 1}$, and its elements are referred to as vectors over $\mathbb{R}_{max}$.

Definition 2.1.4. [6] Let matrix $\boldsymbol{A}$ and vector x, the notation $\boldsymbol{A} \otimes x = \boldsymbol{b}$ is represents a system of maxplus linear equations. A solution to this system is determined by the given vector **x**.

2.2 Supertropical Algebra

Definition 2.2.1. [8] Supertropical algebra is a structure denoted as $(R, \mathcal{G}_0, v\ )$ where R is a semiring with neutral element $\varepsilon = 0_R$ and unity element $e = 1_R$. The set $\mathcal{G}_0 = \mathcal{G} \cup 0_R \subset R$ is called the ghost ideal, and it is associated with a map $v :\ R \to \mathcal{G}_0$, satisfying $v^2 = v$ along with the following condition:

$$a \oplus b = \begin{cases} a, & v(a) \succ v(b) \\ v(a), & v(a) = v(b) \end{cases} \quad (4)$$

Here, the monoid $\mathcal{T}$ represents the tangible elements corresponding to the original max plus algebra. Elements of $\mathcal{G}$ are called ghost elements and are referred to as the ghost map $v :\ R \to \mathcal{G} \cup 0_R$.

We denote $v(a) = a^v$ and define the v-order on R by

$$a \succcurlyeq_v b \leftrightarrow a^v \succcurlyeq b^v \text{ and } a \succ_v b \leftrightarrow a^v \succ b^v \quad (5)$$

Definition 2.2.2. [9] The ghost surpassing relation on R is defined as follows:

$$a \vDash b \text{ if } a = b + g \text{ for some } g \in \mathcal{G}_0 \quad (6)$$

Definition 2.2.3. [9] The partial order relation $\prec$ on $\mathcal{T}$ is defined as following rules:

1. $-\infty \prec a$ for all $a \in \mathcal{T} \setminus -\infty$
2. For any real numbers $a < b$ then $a \prec b$, $a \prec b^v$, $a^v \prec b$ and $a^v \prec b^v$
3. $a \prec a^v$ for every $a \in\ R$

Definition 2.2.4. [10] For any semiring R, $M_n(R)$ represents the semiring of $n \times n$ matrices for every $n \in N$, $n \neq 0$ where matrix addition and multiplication in R. In $M_n(R)$, the identity matrix acts as the unit element. The supertropical determinant of matrices $A \in M_n(R)$ is defined by

$$|A| = \oplus_{\sigma \in S_n} a_1\sigma_1 \dots a_n\sigma_n \tag{7}$$

For a matrix $A \in M_n(R)$, the minor $M_{i,j}$ is defined as the determinant obtained by deleting the i rows and j column of A. The adjoint matrix of A is given by

$$adj(A) = \left(cof(A)\right)^T \text{where } Cof_{i,j} = M_{i,j}. \tag{8}$$

A matrix $A \in M_n(R)$ is called nonsingular if $|A| \in \mathcal{T}$ and singular if $|A| \in \mathcal{G}_0$.

3 Results and Discussion

In this section, we will discuss homogeneous linear systems. To motivate the discussion on homogeneous linear systems, we will consider a specific homogeneous system in the max-plus algebra. The system is defined as follows A⊗x=ε. The selection of matrix A is based on the structure of the supertropical semiring, where tropical addition (⊕) and tropical multiplication (⊗) operations apply. As explained by Izhakian (2024), this structure is crucial in the analysis of matrices and supertropical linear equations because the algebraic properties differ from those of matrices in classical algebra. The matrix A chosen in this study adheres to the fundamental principles of the supertropical semiring, where matrix entries operate under tropical operations to ensure the consistency of solutions and analysis results within the supertropical space [11].

$$A = \begin{bmatrix} 0 & 3 & -\infty \\ 1 & 4 & 6 \\ -\infty & -\infty & 0 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

In matrix multiplication form, it can be written as:

$$\begin{bmatrix} 0 & 3 & -\infty \\ 1 & 4 & 6 \\ -\infty & -\infty & 0 \end{bmatrix} \otimes \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\infty \\ -\infty \\ -\infty \end{bmatrix}$$

The above system is equivalent to

$$(0 \otimes x_1) \oplus (3 \otimes x_2) \oplus (-\infty \otimes x_2) = -\infty$$

$$(1 \otimes x_1) \oplus (4 \otimes x_2) \oplus (6 \otimes x_2) = -\infty$$

$$(-\infty \otimes x_1) \oplus (-\infty \otimes x_2) \oplus (0 \otimes x_2) = -\infty$$

The system of equations $A \otimes x = \varepsilon$ does not have a nontrivial solution. This is because if there were a nontrivial solution, it would imply the existence of a vector $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ Where not all components are equal to ε, the condition for nontrivial solutions is not satisfied.

$$\begin{bmatrix} 0 & 3 & -\infty \\ 1 & 4 & 6 \\ -\infty & -\infty & 0 \end{bmatrix} \otimes \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\infty \\ -\infty \\ -\infty \end{bmatrix}$$

obtained

$$(-\infty \otimes x_1) \oplus (-\infty \otimes x_2) \oplus (0 \otimes x_2) = -\infty \leftrightarrow x_2 = -\infty$$

$$(0 \otimes x_1) \oplus (3 \otimes x_2) \oplus (-\infty \otimes x_2) = -\infty \leftrightarrow x_1 \oplus (3 \otimes x_2) = 3$$

There will be no $x_1, x_2 \in \mathbb{R}_{max}$ such that $-\infty \leftrightarrow x_1 \oplus (3 \otimes x_2) = 3$ Therefore, the only solution to this system is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\infty \\ -\infty \\ -\infty \end{bmatrix}$$

Thus, the equation $A \otimes x = \varepsilon$ only has the trivial solution and does not have any nontrivial solutions. In the previous discussion, it was explained that a special semiring, which is an extension of R_{max}, can be constructed. This allows the solution of linear equation systems to be generalized using the ghost surpass relation in R [12]. Following the discussion from the previous section, using the ghost surpass relation, the solution of the system $A \otimes x = \varepsilon$ is weakened to $A \otimes x \vDash \varepsilon$. Below is an explanation of the ghost surpass relation in supertropical algebra for homogeneous equations.

Definition 3.1. [5]. Given $a \in R$, we have $a \vDash \varepsilon \leftrightarrow a = \varepsilon \oplus c, c \in \mathcal{G}_0$

Definition 3.2. [5]. Given $a \in R$, we have $a \vDash \varepsilon \leftrightarrow a \in \mathcal{G}_0$

Next, we will discuss the solution of a homogeneous equation using the ghost surpass relation in R. For $x \vDash \epsilon$ the solution set of x is given by $\{\varepsilon\} \cup \{b^v, b \in \mathcal{T}\}$.

Lemma 3.1. [13]. If $a \in \mathcal{T}$ and $x \in \mathcal{T}_0$ then the equation $a \otimes x = \mathcal{G}_0$ has only the trivial solution $x = \varepsilon$

Proof.
From Definition 3.1, we know that $a \vDash \varepsilon \leftrightarrow a = \varepsilon \oplus c, c \in \mathcal{G}_0$.
Based on this, we can conclude that if $a \in \mathcal{T}$, then for every $x \in \mathcal{T}_0$, there is only the trivial solution $x = \varepsilon$ such that $a \otimes x \in \mathcal{G}_0$.

Next, we will extend the ghost surpass relation to vectors in the context of homogeneous linear equations in supertropical algebra.

Definition 3.3. Given $u \in R^n$, then $u \vDash \varepsilon$ if and only if $u \in {\mathcal{G}_0}^n$, which is equivalent to $u_i \vDash \varepsilon$ if and only if $u_i \in \mathcal{G}_0$ for every $i \in n$.

Definition 3.4. Given $A \in M_n(R)$, and $u \in R^n$ then the system $A \otimes x \vDash \varepsilon$ if and only if $A \otimes x \vDash {\mathcal{G}_0}^n$

The following are some definitions related to the solution $A \otimes x \vDash {\mathcal{G}_0}^n$

Definition 3.5. [9] A set of vectors $V = \{v_1, v_2, \dots, v_n\} \in R^n$ is called supertropical linearly independent if

$$\oplus_{i=1}^{n} \alpha_i \otimes v_i \in {\mathcal{G}_0}^n$$

implies that $\alpha_i = \varepsilon = -\infty$ for every $i \in n$.

Definition 3.6. [10] A set of vectors $V = \{v_1, v_2, \dots, v_n\} \in R^n$ It is called supertropical linearly dependent if there exist scalars. $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathcal{T}_0$ where not all $\alpha_i = \varepsilon$ such that

$$\oplus_{i=1}^{n} \alpha_i \otimes v_i \in {\mathcal{G}_0}^n$$

with $\mathcal{T}_0 \in \mathcal{T} \cup \{-\infty\}$ and $i \in n$.

The following examples illustrate the concepts related to Definitions 3.5 and 3.6.

Example 3.1. Given $V = \{v_1, v_2, v_3\}$ in R^3 with $v_1 = \begin{bmatrix} 0 \\ -\infty \\ -\infty \end{bmatrix}$, $v_2 = \begin{bmatrix} 20 \\ 5 \\ -\infty \end{bmatrix}$, and, $v_3 = \begin{bmatrix} -\infty \\ 4 \\ 0 \end{bmatrix}$. The vectors v_1, v_2, v_3 are supertropical linearly independent. To prove this, it must be shown that the only way for

$$\propto_1 \otimes \begin{bmatrix} 0 \\ -\infty \\ -\infty \end{bmatrix} \oplus \propto_2 \otimes \begin{bmatrix} 20 \\ 5 \\ -\infty \end{bmatrix} \oplus \propto_3 \otimes \begin{bmatrix} -\infty \\ 4 \\ 0 \end{bmatrix} \in {\mathcal{G}_0}^3$$

That is, if all scalars. $\propto_1, \propto_2, \propto_3$ are $-\infty$. The above equation can be written as a linear system with variables $\propto_1, \propto_2, \propto_3$ as follows.

$$(0\otimes \propto_1)\oplus(20\otimes \propto_2)\oplus(-\infty\otimes \propto_3) \in \mathcal{G}_0$$

$$(-\infty\otimes \propto_1)\oplus(5\otimes \propto_2)\oplus(4\otimes \propto_3) \in \mathcal{G}_0$$

$$(-\infty\otimes \propto_1)\oplus(-\infty\otimes \propto_2)\oplus(0\otimes \propto_3) \in \mathcal{G}_0$$

In matrix multiplication form, it can be written as:

$$\begin{bmatrix} 0 & 20 & -\infty \\ -\infty & 5 & 4 \\ -\infty & -\infty & 0 \end{bmatrix} \otimes \begin{bmatrix} \propto_1 \\ \propto_2 \\ \propto_3 \end{bmatrix} = \begin{bmatrix} -\infty \\ -\infty \\ -\infty \end{bmatrix}$$

Obtained,

$$(-\infty\otimes \propto_1)\oplus(-\infty\otimes \propto_2)\oplus(0\otimes \propto_3) \in \mathcal{G}_0 \leftrightarrow \propto_1 = \propto_3$$

It can be seen that a scalar exists. $\propto_1 = \propto_3$, so for every scalar $\propto_1 = \propto_3 \in \mathcal{T}_0$ where $\propto_1 = \propto_3$, the equation will be satisfied. Therefore, this system has a nontrivial solution. From this example, it can be seen that the coefficient matrix of this system is singular. This can be demonstrated as follows.

$$|V| = \begin{vmatrix} 0 & 20 & -\infty \\ -\infty & 5 & 4 \\ -\infty & -\infty & 0 \end{vmatrix} = 5 \in \mathcal{T}$$

The determinant will be calculated.

$$|V| = (v_{11}\otimes v_{22}\otimes v_{33}) \oplus (v_{11}\otimes v_{23}\otimes v_{32}) \oplus (v_{12}\otimes v_{21}\otimes v_{33}) \oplus (v_{12}\otimes v_{23}\otimes v_{31}) \oplus (v_{13}\otimes v_{22}\otimes v_{31}) \oplus (v_{13}\otimes v_{22}\otimes v_{31})$$

$$|V| = (0\otimes 5\otimes 0) \oplus (0\otimes 4\otimes -\infty) \oplus (20\otimes -\infty\otimes 0) \oplus (20\otimes 4\otimes -\infty) \oplus (-\infty\otimes 5\otimes -\infty) \oplus (-\infty\otimes 5\otimes -\infty)$$

$$|V| = 5 \oplus 4 \oplus 2 \oplus -\infty \oplus -\infty \oplus -\infty = 5$$

Since $|V| = 5 \in \mathcal{T}$, the matrix V is nonsingular.

Example 3.2. Given $V = \{v_1, v_2, v_3\}$ in R^3 with $v_1 = \begin{bmatrix} 0 \\ -\infty \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 3 \\ 4 \\ -\infty \end{bmatrix}$, and, $v_3 = \begin{bmatrix} \infty \\ 5 \\ 0 \end{bmatrix}$, the vectors v_1, v_2, v_3 are supertropically dependent. To prove this, it must be shown that there exist scalars $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{T}_0$, where not all $\alpha_i = \varepsilon$ for $i = 1,2,3$. The equation is

$$\alpha_1 \otimes \begin{bmatrix} 0 \\ -\infty \\ 0 \end{bmatrix} \oplus \alpha_2 \otimes \begin{bmatrix} 3 \\ 4 \\ -\infty \end{bmatrix} \oplus \alpha_3 \otimes \begin{bmatrix} \infty \\ 5 \\ 0 \end{bmatrix} \in \mathcal{G}_0{}^3$$

The above equation can be written as a linear system with variables $\propto_1, \propto_2, \propto_3$ as follows.

$$(0\otimes \propto_1)\oplus(3\otimes \propto_2)\oplus(-\infty\otimes \propto_3) \in \mathcal{G}_0$$

$$(-\infty\otimes \propto_1)\oplus(4\otimes \propto_2)\oplus(5\otimes \propto_3) \in \mathcal{G}_0$$

$$(0\otimes \propto_1)\oplus(-\infty\otimes \propto_2)\oplus(0\otimes \propto_3) \in \mathcal{G}_0$$

In matrix form, this system can be expressed as follows:

$$\begin{bmatrix} 0 & 3 & -\infty \\ -\infty & 4 & 5 \\ 0 & -\infty & 0 \end{bmatrix} \otimes \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \in \mathcal{G}_0{}^3$$

Obtained,

$$(0\otimes \propto_1)\oplus(-\infty\otimes \propto_2)\oplus(0\otimes \propto_3) \in \mathcal{G}_0 \leftrightarrow \propto_1 = \propto_3$$

It can be seen that there exists a scalar $\propto_1 = \propto_3$, so for every scalar $\propto_1 = \propto_3 \in \mathcal{T}_0$ where $\propto_1 = \propto_3$, the equation will be satisfied. Therefore, this system has a nontrivial solution. From this example, it can be seen that the coefficient matrix of this system is singular. This can be demonstrated as follows.

$$|V| = \begin{vmatrix} 0 & 3 & -\infty \\ -\infty & 4 & 5 \\ 0 & -\infty & 0 \end{vmatrix} = 5 \in \mathcal{T}$$

The determinant will be calculated.

$$|V| = (v_{11}\otimes v_{22}\otimes v_{33}) \oplus (v_{11}\otimes v_{23}\otimes v_{32}) \oplus (v_{12}\otimes v_{21}\otimes v_{33}) \oplus (v_{12}\otimes v_{23}\otimes v_{31}) \oplus (v_{13}\otimes v_{22}\otimes v_{31}) \oplus (v_{13}\otimes v_{22}\otimes v_{31})$$

$$|V| = (0\otimes 4\otimes 0) \oplus (0\otimes 5\otimes -\infty) \oplus (3\otimes -\infty\otimes 0) \oplus (3\otimes 5\otimes 0) \oplus (3\otimes 5\otimes 0) \oplus (-\infty\otimes 4\otimes 0)$$

$$|V| = 4 \oplus -\infty \oplus -\infty \oplus 8 \oplus 8 \oplus -\infty = 8^v$$

Since $|V| = 8^v \in \mathcal{G}_0$, the matrix V is singular.

The following is an explanation of several concepts that have been discussed. The vectors $\propto_i$ with $i = 1, 2, 3, \ldots, n$ in the supertropical vector space V are linearly independent, which is equivalent to:

$$(x_1 \otimes \propto_1) \oplus (x_2 \otimes \propto_2) .. \oplus (x_3 \otimes \propto_3) \mathcal{G}_0{}^n$$

is satisfied only when $x_1 = x_2 = \cdots = x_n = -\infty$. If V = Rn, then the vectors $\propto_i$ with $i = 1, 2, 3, \ldots, n$ in the vector space V over R being supertropically independent means that the system of homogeneous linear equations.

$$(x_1 \otimes \propto_1) \oplus (x_2 \otimes \propto_2) .. \oplus (x_3 \otimes \propto_3) \mathcal{G}_0{}^n$$

has only the trivial solution $x_i = -\infty$ with $i = 1, 2, 3, \ldots, n$.

If this homogeneous equation has a nontrivial solution, i.e., $x_i \neq -\infty$ for some i with $x_i \in \mathcal{T}_0$. This means that the vectors $\propto_i$ are not supertropically independent or are supertropically dependent.

The following presents the theorem regarding the existence and uniqueness of the solution $A \otimes \boldsymbol{x} \vDash \varepsilon$ in supertropical algebra.

Theorem 3.1. Given $A \in M_n(R)$, the system of equations $A \otimes \boldsymbol{x} \vDash \varepsilon$ has a nontrivial solution if and only if $|A| \in \mathcal{G}_0 \neq \varepsilon$.

Theorem 3.2. Given $A \in M_n(R)$, the system of equations $A \otimes \boldsymbol{x} \vDash \varepsilon$ has a trivial solution if and only if $|A| \in \mathcal{T}_0$.

Next, we discuss the nontrivial solution of the equation $A \otimes \boldsymbol{x} \vDash \varepsilon$ in supertropical algebra.

Proposition 3.1 [8]. Given $A \in M_n(R)$, where $|A| \in \mathcal{G}_0 \neq \varepsilon$ and $x \in \mathcal{T}_0^n$, the system of equations $A \otimes \boldsymbol{x} \vDash \varepsilon$ has a solution $k \otimes x \in R^n$, where $\boldsymbol{x}$ is the $i - th$ column of $adj\ (A)$ for some $i \in n$ and $k \in \mathcal{T}$.

The following examples demonstrate the application of the existence and uniqueness theorems for solutions to the system of equations $A \otimes \boldsymbol{x} \vDash \varepsilon$ in supertropical algebra. These examples aim to show how the theorems can be applied to various cases involving matrix equations.

Example 3.3. Consider the following system of equations in supertropical algebra $A \otimes \boldsymbol{x} \vDash \varepsilon$ with

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 1 & 0 & 5 \\ 2 & 1 & 2 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Solution:
First, calculate the determinant of the matrix A:

$$|A| = \begin{vmatrix} 1 & 3 & -1 \\ 1 & 0 & 5 \\ 2 & 1 & 2 \end{vmatrix} = 10$$

Based on the calculations, it was found that the determinant of matrix A is $|A| = 10 \in \mathcal{T}$, which represents the set of nonsingular elements. According to Theorem 4.6, if $|A| \in \mathcal{T}$, then the system of equations $A \otimes \boldsymbol{x} \vDash \varepsilon$ has only a trivial solution.

Hence, we can conclude that this system does not have a nontrivial solution, and the solution is

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \varepsilon \\ \varepsilon \\ \varepsilon \end{bmatrix}$$

This indicates that the system is singular, and the only solution that satisfies the equation is the trivial.

Example 3.4. Consider the following system of equations in supertropical algebra $A \otimes \boldsymbol{x} \vDash \varepsilon$ with

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 5 \\ 2 & 2 & 2 \end{bmatrix}, \; x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Solution:
First, calculate the determinant of the matrix A:

$$|A| = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 5 \\ 2 & 2 & 2 \end{bmatrix} = 9^{\nu} \in \mathcal{G}_0$$

Based on the calculations, it was found that the determinant of matrix A is $|A| = 9^{\nu} \in \mathcal{G}_0$ which represents the set of singular elements. According to Theorem 4.5, if $|A| \in \mathcal{T}$, then the system of equations $A \otimes \boldsymbol{x} \vDash \varepsilon$ has a nontrivial solution. Therefore, we can conclude that this system has a nontrivial solution, and the solution is

$$x = k \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \text{ for every } k \in \mathcal{T}$$

where $p_1 = 7, p_2 = 7, and\ p_3 = 5$ or $p_1 = 6, p_2 = 6, and\ p_3 = 4$.

4 Conclusion

The existence and uniqueness of the solution of homogeneous linear system $A \otimes \boldsymbol{x} \vDash \varepsilon$ in supertropical algebra can be explained through the determinant properties of matrix A. The system has a nontrivial solution if and only if the determinant $|A| \in \mathcal{G}_0 \neq \varepsilon$. If $|A| \in \mathcal{T}_0$, the system only has a trivial solution where all components of the vector $\boldsymbol{x}$ are zero. Therefore, the determinant of the matrix determines whether the system has only a unique trivial solution or if a nontrivial solution exists, which governs the existence and uniqueness of the solution in supertropical algebra.

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