Contemporary Mathematics and Applications
Vol. 7, No. 2, 2025, pp. 106-116

Numerical Invariants Of Nilpotent Graphs In Integer Modulo Rings

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Abstract. Graph theory offers a robust framework for examining algebraic structures, especially rings and their elements. This paper focuses on the nilpotent graph of rings of the form $\mathbf{Z}_{p^k}$, where p is a prime and $k \in \mathbf{N}$, investigating both their structural and numerical properties. We begin by characterizing the nilpotent elements in these rings and examining their relationship to ring ideals. The study then presents theoretical results on key graph invariants, including connectivity, chromatic number, clique number, and specific subgraph configurations. To complement these, we also analyze numerical invariants such as edge count and degree distribution, which reveal deeper connections between ring-theoretic and graph-theoretic properties. Our results highlight consistent structural patterns in nilpotent graphs of $\mathbf{Z}_{p^k}$ and provide a concrete contribution to algebraic graph theory by bridging properties of commutative rings and their associated graphs.

1 Introduction

Graph theory has become an essential tool in various branches of mathematics, particularly in understanding algebraic structures. One notable application is the study of graphs associated with rings, where elements of a ring serve as vertices, and edges are defined based on specific algebraic properties. Among the many graph constructions in ring theory, the nilpotent graph has gained significant attention due to its deep connections with ideal structures and radical properties of rings [1].

The concept of nilpotent elements in a ring plays a crucial role in determining the ring's behavior and structure. A nilpotent element refers to an element whose power equals zero for some positive integer exponent.. The collection of such elements forms the nilradical of a ring, which is essential in both commutative and non-commutative algebra. By constructing a graph in which vertices represent nilpotent elements and adjacency is determined by multiplication, one can visualize and analyze these algebraic properties using graph-theoretic approaches. By constructing a graph in which vertices represent nilpotent elements and adjacency is determined by multiplication, one can visualize and analyze these algebraic properties through graph-theoretic approaches [2]. Some applications of this concept can be found in various fields, such as computer networking

Received: February 10, 2025
Accepted: August 14, 2025

and molecular graphs, which play a crucial role in chemical structure analysis [3] and communication system optimization [4].

This paper explores the structural properties of nilpotent graphs associated with integer rings modulo prime power orders. We begin by characterizing the nilpotent elements in these rings and identifying patterns that emerge in their distribution. Several theorems are presented to establish relationships between nilpotent elements and ideals within the ring. Furthermore, we introduce the concept of the nilpotent graph, analyzing its connectivity, chromatic properties, clique structure, and subgraph formations [5].

Through this study, we demonstrate how nilpotent graphs can serve as a tool to visualize and analyze structural properties of rings, particularly highlighting the behavior of nilpotent elements and their relation to ideals in $\mathbb{Z}_{p^k}$. The results contribute to the development of algebraic graph theory by providing graph-theoretic characterizations of ring-theoretic concepts. Furthermore, this work paves the way for future research on classifying ring elements using graph invariants, exploring connections between ideal structures and subgraphs, and developing computational methods for analyzing algebraic objects via their graph representations.

2 Literature review

Numerical invariants of a graph representation of a group provide essential insights into the algebraic structure it represents. Nurhabibah et al. give a detailed analysis of the numerical invariants of the coprime graph of a generalized quaternion group. They characterize the coprime elements within this group and examine their structural properties. Various graph invariants, such as connectivity, chromatic number, clique structure, and degree distribution, are analyzed to identify emerging patterns. The study demonstrates how the coprime graph provides a novel perspective on group structures through a graph-theoretic approach. The findings contribute to the field of algebraic graph theory and open new avenues for further research on graph representations in algebraic systems [6].

In 2023 Malik et al. studied nilpotent graphs in algebraic structures, focusing on the ring of integers modulo a prime power. They show that for $\mathbb{Z}_p$, the nilpotent graph forms a star graph $K_{1,p-1}$,, with zero as the central node. For $\mathbb{Z}_{p^k}$, the nilpotent elements form a complete subgraph K_{p^k-1} and multiple star subgraphs. Their findings align with previous work by Nikmehr and Khojasteh and expand understanding in algebraic graph theory. The study suggests applications in computational algebra and cryptography [1].

In 2024, Malik et al. further investigated the chemical topological properties of nilpotent graphs in modular rings of prime power order. They analyzed topological indices such as the Wiener, Zagreb, and Gutman indices to describe molecular structures mathematically.

Their study demonstrated how these indices reflect the connectivity and adjacency of nilpotent elements. The findings provide deeper insight into the interplay between algebraic structures and graph-based chemical models [7].

Based on the work of Malik et al. in 2023 and 2024, this research aims to find its numerical invariants in nilpotent graphs. Following the approach of Nurhabibah et al., who investigated numerical invariants in related algebraic graph structures, we aim to explore additional graph parameters such as chromatic number, clique number, and diameter. These numerical invariants could offer further structural insights and potential applications in both algebraic graph theory and computational modeling.

The study of nilpotent elements in rings has been a fundamental topic in algebra, particularly in understanding ring structures and their ideal properties. Various researchers have explored the role of nilpotent elements in different classes of rings, including their influence on radical theories and algebraic graph representations. One key approach in analyzing these elements is through the construction of graphs that capture the multiplicative interactions among nilpotent elements within a ring.

Definition 1. [8] Let $\boldsymbol{R}$ be a ring. An element $\boldsymbol{r}$ in $\boldsymbol{R}$ is called a nilpotent element if $\boldsymbol{r^k} = \boldsymbol{0_R}$, for $\boldsymbol{k} \in \mathbb{N}$.

Definition 2. [8] Let $N(R)$ be a non-empty subset of the ring $(R, +,\cdot)$. The set $N(R)$ is called a nilpotent set if every element of $N(R)$ is a nilpotent element of $(R, +,\cdot)$.

Based on this, the pattern of nilpotent sets in the ring $\mathbb{Z}_{\boldsymbol{n}}$ is obtained for $\boldsymbol{n} = \boldsymbol{p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}}$ as explained in the following theorem.

Theorem 1 [7] Let $\mathbb{Z}_n$ be a ring and $n = p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$ with p_i prime and $k_i \in \mathbb{N}$, $i = 1,2, \dots, m$. Hence, for $\bar{x} \in \mathbb{Z}_n$, we have $\bar{x} \in N(\mathbb{Z}_n)$ if and only if $\bar{x} = \overline{q(p_1 p_2 \dots p_m)}$ with $q \in \mathbb{N}$.

For example, the ring $\mathbb{Z}_{4608}$ where $\mathbb{Z}_{4608} = \mathbb{Z}_{2^3.3^2.4^2} = \{\bar{0}, \bar{1}, \dots, \overline{4607}\}$, we have $\overline{240} \in N(\mathbb{Z}_{4608})$ since $\overline{240} = \overline{10(2.3.4)}$. Building upon Theorem 1, we can determine the cardinality of all nilpotent elements, as articulated in the following theorem.

Theorem 2 [7] The cardinality of all nilpotent element of $\mathbb{Z}_n$ for $n = p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$ is:

$$|N(\mathbb{Z}_n)| = p_1^{k_1 - 1} p_2^{k_2 - 1} \dots p_m^{k_m - 1}$$

Graphs provide a useful way to visualize algebraic structures, including rings and their elements. In particular, nilpotent elements play a crucial role in ring theory, influencing ideals and radical properties. By representing these elements as vertices and defining adjacency based on their interactions, we obtain a nilpotent graph, which offers insights into the structure of the ring. The following definition formalizes this concept.

Definition 3. [9] For a ring R, the nilpotent graph of R, denoted by $\mathbf{\Gamma_R}$ is a graph whose vertex set is $\boldsymbol{R}$ for $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{R}$ are said to be adjacency if $\boldsymbol{uv} \in \boldsymbol{N}(\boldsymbol{R})$, where $\boldsymbol{N}(\boldsymbol{R})$ is the nilpotent set of ring $\boldsymbol{R}$.

Theorem 3. [7] If $\mathbb{Z}_{\boldsymbol{n}}$ is a modulo integer ring with $\boldsymbol{n} = \boldsymbol{p^k}$where $\boldsymbol{k} \in \mathbb{N}$, then there exists a subgraph of $\mathbf{\Gamma_N}(\mathbb{Z}_{\boldsymbol{n}})$which is a complete subgraph of $\boldsymbol{K_{p^{k-1}}}$.

To illustrate this, we give an example for the ring $\mathbb{Z}_9$

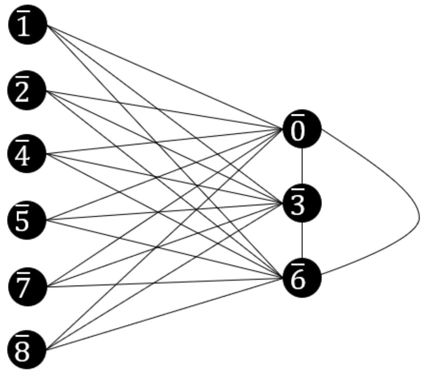

Figure 2 Graph of $\boldsymbol{\Gamma_N}(\mathbb{Z}_{3^2})$

In the graph $\mathbf{\Gamma_N}(\mathbb{Z}_{3^2})$ the nilpotent vertices are $\bar{\mathbf{0}}, \bar{\mathbf{3}},$ and $\bar{\mathbf{6}}$, all of which are pairwise adjacent. As a result, these vertices form a complete subgraph, denoted by $\boldsymbol{K_3}$. This is consistent with the theorem stating that the nilpotent graph of $\mathbb{Z}_{2^2}$ forms a complete graph $\boldsymbol{K_{2^2-1}}$, as shown below

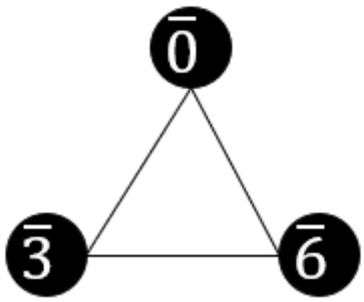

Figure 3 Complete Subgraph $\boldsymbol{\Gamma_N}(\mathbb{Z}_{3^2})$

Another characteristics of nilpotent graph on the integer ring modulo with an arbitrary prime orde is the existence of a twin star subgraph

Theorem 4. [7] If $\mathbb{Z}_n$ is the integer ring modulo with $\boldsymbol{n = p^k}$ where $\boldsymbol{k} \in \mathbb{N}$, then $\boldsymbol{\Gamma_N}(\mathbb{Z}_n)$ contains $\boldsymbol{p^{k-1}}$ star subgraph $\boldsymbol{K_{1,n-1}}$.

It is easy to see that the star subgraph $\boldsymbol{K_{1,n-1}}$ is the largest star subgraph of the graph $\boldsymbol{\Gamma_N}(\mathbb{Z}_{p^k})$. Based on Figure 2, the nilpotent graph $\boldsymbol{\Gamma_N}(\mathbb{Z}_{3^2})$ has three star subgraphs $\boldsymbol{K_{1,8}}$.

Besides subgraphs, there are other characteristics of the nilpotent graph of the ring of integers modulo $\boldsymbol{n}$ with an arbitrary prime power order, as stated in the following theorem.

Theorem 5. [7] A nilpotent graph of the ring $\mathbb{Z}_n$ for $\boldsymbol{n = p^k}$ where $\boldsymbol{p}$ is prime and $\boldsymbol{k} \in \mathbb{N}$, denoted by $\boldsymbol{\Gamma_N}(\mathbb{Z}_n)$, has an edge count (size) given by:

$$\left|E\left(\Gamma_N(\mathbb{Z}_{p^k})\right)\right| = \left(p^{2k-2}(p-1)\right) + \frac{(p^{2k-2} - p^{k-1})}{2}$$

In graph theory, **graph coloring** refers to the assignment of colors to elements of a graph Γ, such as its vertices $V(\Gamma)$, edges $E(\Gamma)$ or both $V(\Gamma) \cup E(\Gamma)$. If the coloring applies only to the vertices, it is called **vertex coloring**. The chromatic number is the smallest number of colors needed to color the vertices so that no two adjacent vertices have the same color.

Definition 4. [10] The chromatic number **of a graph Γ is the minimum number of colors needed in any proper vertex coloring of Γ, such that adjacent vertices have different colors. It is denoted by $\chi(\Gamma)$.**

The nilpotent graph of the ring $\mathbb{Z}_5$ has a minimal coloring. The vertex 0 is adjacent to all other vertices, requiring a different color from the remaining four vertices. Moreover, any two vertices other than $\bar{0}$ are not adjacent, allowing them to be colored the same. Thus, the minimal number of colors needed is 2, meaning that the chromatic number is $\chi(\Gamma_N(\mathbb{Z}_5)) = 2$.

3 Results and Discussion.

This chapter presents the main findings from the analysis of nilpotent graphs in integer modulo rings. It highlights key structural and numerical properties, such as chromatic number, degree, and clique size, supported by examples and figures to illustrate the results clearly

Theorem 6. Let $\boldsymbol{\Gamma_N}(\mathbb{Z}_n)$ is the nilpotent graph of the ring $\mathbb{Z}_n$. If $\boldsymbol{n = p^k}$ where $\boldsymbol{p}$ is a prime and $\boldsymbol{k} \in \boldsymbol{N}$, then the chromatic number of $\boldsymbol{\Gamma_N}(\mathbb{Z}_n)$ is

$$\boldsymbol{\chi\left(\Gamma_N(\mathbb{Z}_{p^k})\right) = p^{k-1} + 1}$$

Proof. We will show that $\boldsymbol{\chi\left(\Gamma_N(\mathbb{Z}_{p^k})\right) = p^{k-1} + 1}$. Based on **Theorem 3**, the graph

$\boldsymbol{\Gamma_N(Z_{p^k})}$ contains a complete subgraph $\boldsymbol{K_{p^{k-1}}}$, every vertex in this subgraph is adjacent to every other vertex because they are nilpotent vertices. Therefore, at least $\boldsymbol{p^{k-1}}$ colors are needed to color the complete subgraph $\boldsymbol{K_{p^{k-1}}}$.

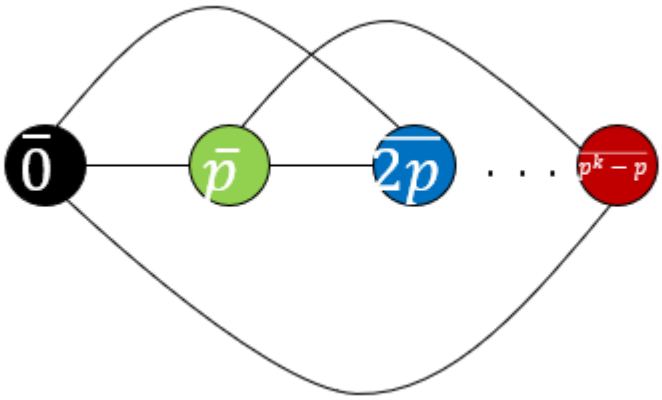

Figure 7 Complete Subgraph $\boldsymbol{K_{p^{k-1}}}$

A non-nilpotent vertex is adjacent to all nilpotent vertices but not to other non-nilpotent vertices. Therefore, each non-nilpotent vertex can be colored with a single color distinct from those assigned to the nilpotent vertices. It follows that, So number of colors needed to color the vertices so that no two adjacent vertices have the same color is $p^{k-1} + 1$.

Suppose p^{k-1} colors are sufficient to color $\Gamma_N(\mathbb{Z}_{p^k})$. Since there are p^{k-1} nilpotent vertices, all of which are adjacent to every other vertex, these colors would be entirely used for the nilpotent vertices. The non-nilpotent vertices would then remain uncolored, making it impossible to properly color the entire graph with only p^{k-1} colors. Thus, $p^{k-1} + 1$ is the smallest number of colors needed to color the vertices so that no two adjacent vertices have the same color. From this, we conclude that:

$$\chi\left(\Gamma_N(\mathbb{Z}_{p^k})\right) = \chi(K_{p^{k-1}}) + 1 = p^{k-1} + 1 \blacksquare.$$

To illustrate Theorem 6, the graph $\Gamma_N(\mathbb{Z}_{3^2})$ in **Figure 2** contains a complete subgraph K_3, meaning its chromatic number is $\chi(K_3) = 3^{2-1} = 3$ with one additional color assigned to all non-nilpotent vertices to ensure they are distinct from the nilpotent vertices. Thus, we have $\chi(\Gamma_N(\mathbb{Z}_{3^2}) = 3^{2-1} + 1 = 4$

Theorem 7. Let $\bar{v} \in V\left(\Gamma_N(\mathbb{Z}_{p^k})\right)$, then:

i. If $\bar{v} \in N(\mathbb{Z}_{p^k})$ then $deg(\bar{v}) = p^k - 1$

ii. If $\bar{v} \notin N(\mathbb{Z}_{p^k})$ then $deg(\bar{v}) = p^{k-1}$

Proof.

i. For every $\boldsymbol{\bar{v}} \in \boldsymbol{V}\left(\boldsymbol{\Gamma_N}(\mathbb{Z}_{p^k})\right)$ such that $\boldsymbol{\bar{v}} \in \boldsymbol{N}(\mathbb{Z}_{p^k})$.

According to **Theorem 1**, nilpotent vertices are adjacent to all other vertices in the graph $\Gamma_N(\mathbb{Z}_{p^k})$, as evident. Since the total number of vertices is p^k, the degree of a nilpotent vertex is $p^k - 1$

ii. For every $\bar{v} \in V\left(\Gamma_N(\mathbb{Z}_{p^k})\right)$ such that $\bar{v} \notin N(\mathbb{Z}_{p^k})$.

From **Theorem 1**, $\bar{v}$ is adjacent to all nilpotent vertices of the ring $\mathbb{Z}_{p^k}$. This means $\bar{v}$ is adjacent to $p^{(k-1)}$ vertices in the graph. However, according to Theorem 1, $\bar{v}$ is not adjacent to any non-nilpotent vertices, meaning it is adjacent only to nilpotent vertices. Thus, the degree of $\bar{v}$ is $p^{(k-1)}$ ■.

To illustrate Theorem 7, consider the nilpotent graph of the ring $\mathbb{Z}_{5^2}$ are as follows.

i. For vertices that are nilpotent elements of $\mathbb{Z}_{5^2}$ the degree is given by:

$$5^2 - 1 = 24$$

ii. For vertices that are nilpotent elements of $\mathbb{Z}_{5^2}$ the degree is given by:

$$5^{2-1} = 5$$

Definition 5. A **clique** in a graph Γ is a subset $V'(\Gamma)$ of the vertex set $V(\Gamma)$ such that in the induced subgraph $\Gamma'[V']$, every pair of distinct vertices in $V'(\Gamma)$ is adjacent. The **clique number** of Γ denoted by $\omega(\Gamma)$ is the size of the largest clique in Γ.

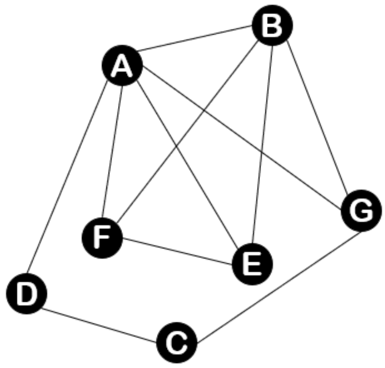

Figure 8 Graph of $\boldsymbol{\Gamma_5}$

In this graph, the largest clique can be determined as follows:

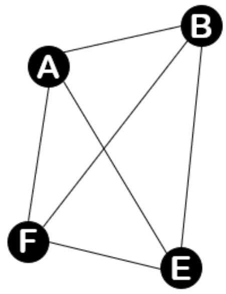

Figure 9 Subgraph of $\boldsymbol{\Gamma_5'}$

Thus, the **clique number** of the graph Γ_5 is given by:

$$\omega(\Gamma_5) = 4$$

Theorem 8. The clique number of the nilpotent graph $\Gamma_N(\mathbb{Z}_{p^k})$ is given by:

$$\omega\left(\Gamma_N(\mathbb{Z}_{p^k})\right) = p^{k-1} + 1.$$

Proof. According to Theorem 3, there are exactly $\boldsymbol{p^{k-1}}$ nilpotent vertices, all of which are adjacent to each other. Furthermore, each non-nilpotent vertex is adjacent to every nilpotent vertex but not to any other non-nilpotent vertices. It follows that precisely one non-nilpotent vertex can be chosen to form a complete subgraph of maximum cardinality.

As a result, the largest complete subgraph is $K_{p^{k-1}+1}$ in terms of cardinality. Therefore, the clique number is $\omega\left(\Gamma_N(\mathbb{Z}_{p^k})\right) = p^{k-1} + 1,$ this completes the proof ■.

For examples, the clique number of the nilpotent graph of the ring $\mathbb{Z}_{5^2}$ is given below:

$$\omega\left(\Gamma_N(\mathbb{Z}_{5^2})\right) = 5^{2-1} + 1 = 6.$$

Corollary 1. For $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}\left(\boldsymbol{\Gamma_N}(\mathbb{Z}_{p^k})\right)$, if $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{N}(\mathbb{Z}_{p^k})$ then $(\boldsymbol{u}, \boldsymbol{v}) \in \boldsymbol{E}\left(\boldsymbol{\Gamma_N}(\mathbb{Z}_{p^k})\right)$ and $\boldsymbol{d}(\boldsymbol{u}, \boldsymbol{v}) = \mathbf{1}$

Proof. By Theorem 1, let $\boldsymbol{u} = \overline{\boldsymbol{rp}}$ and $\boldsymbol{v} = \overline{\boldsymbol{sp}}$ where $\boldsymbol{r}, \boldsymbol{s} \in \mathbb{N}$, then $\boldsymbol{uv} = \overline{(\boldsymbol{rp})(\boldsymbol{sp})} = \overline{(\boldsymbol{rps})\boldsymbol{p}}$ where $\boldsymbol{rps} \in \mathbb{N}$. Thus $\boldsymbol{uv} \in \boldsymbol{N}(\mathbb{Z}_{p^k})$ By the Definition 1, we conclude that $(\boldsymbol{u}, \boldsymbol{v}) \in \boldsymbol{E}\left(\boldsymbol{\Gamma_N}(\mathbb{Z}_{p^k})\right)$ and the distance between $\boldsymbol{u}$ and $\boldsymbol{v}$ is $\boldsymbol{d}(\boldsymbol{u}, \boldsymbol{v}) = \mathbf{1}$.

Corollary 2. For $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}\left(\boldsymbol{\Gamma_N}(\mathbb{Z}_{p^k})\right)$, if $\boldsymbol{u} \in \boldsymbol{N}(\mathbb{Z}_{p^k})$ $\boldsymbol{v} \notin \boldsymbol{N}(\mathbb{Z}_{p^k})$ then $(\boldsymbol{u}, \boldsymbol{v}) \in \boldsymbol{E}\left(\boldsymbol{\Gamma_N}(\mathbb{Z}_{p^k})\right)$ and $\boldsymbol{d}(\boldsymbol{u}, \boldsymbol{v}) = \mathbf{1}$

Proof. By Theorem 1, let $\boldsymbol{u} = \overline{\boldsymbol{rp}}$ and $\boldsymbol{v} = \overline{\boldsymbol{x}} \neq \overline{\boldsymbol{sp}}$ where $\boldsymbol{r}, \boldsymbol{s} \in \mathbb{N}$, then $\boldsymbol{uv} = \overline{(\boldsymbol{rp})(\boldsymbol{x})} = \overline{(\boldsymbol{rx})\boldsymbol{p}}$ where $\boldsymbol{rps} \in \mathbb{N}$. Thus $\boldsymbol{rx} \in \boldsymbol{N}(\mathbb{Z}_{p^k})$ By the Definition 1, we conclude that $(\boldsymbol{u}, \boldsymbol{v}) \in \boldsymbol{E}\left(\boldsymbol{\Gamma_N}(\mathbb{Z}_{p^k})\right)$ and the distance between $\boldsymbol{u}$ and $\boldsymbol{v}$ is $\boldsymbol{d}(\boldsymbol{u}, \boldsymbol{v}) = \mathbf{1}$.

Corollary 3. For $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}\left(\boldsymbol{\Gamma_N}(\mathbb{Z}_{p^k})\right)$, if $\boldsymbol{u}, \boldsymbol{v} \notin \boldsymbol{N}(\mathbb{Z}_{p^k})$ then $(\boldsymbol{u}, \boldsymbol{v}) \notin \boldsymbol{E}\left(\boldsymbol{\Gamma_N}(\mathbb{Z}_{p^k})\right)$ and $\boldsymbol{d}(\boldsymbol{u}, \boldsymbol{v}) = \mathbf{2}$

Proof. By Theorem 1, let $\boldsymbol{u} \neq \overline{\boldsymbol{rp}}$ and $\boldsymbol{v} \neq \overline{\boldsymbol{sp}}$ where $\boldsymbol{r}, \boldsymbol{s} \in \mathbb{N}$. Since $\boldsymbol{p}$ is prime, it follows that $\boldsymbol{uv} \neq \overline{\boldsymbol{tp}}$, implying $(\boldsymbol{u}, \boldsymbol{v}) \notin \boldsymbol{E}\left(\boldsymbol{\Gamma_N}(\mathbb{Z}_{p^k})\right)$ For $\boldsymbol{w} \in \boldsymbol{N}(\mathbb{Z}_{p^k})$, we have $(\boldsymbol{w}, \boldsymbol{u}), (\boldsymbol{w}, \boldsymbol{v}) \in \boldsymbol{E}\left(\boldsymbol{\Gamma_N}(\mathbb{Z}_{p^k})\right)$, meaning that there exists a path from $\boldsymbol{u}$ to $\boldsymbol{v}$ given by:

$$u \to w \to v$$

Thus, the distance between u and v is:

$$d(u, v) = 2.$$

Theorem 9. Let $\bar{v} \in V\left(\Gamma_N(\mathbb{Z}_{p^k})\right)$, then:

i. If $\bar{v} \in N(\mathbb{Z}_{p^k})$ then $deg(\bar{v}) = p^k - 1$

ii. If $\bar{v} \notin N(\mathbb{Z}_{p^k})$ then $deg(\bar{v}) = p^{k-1}$

Proof.

i. For every $\bar{v} \in V\left(\Gamma_N(\mathbb{Z}_{p^k})\right)$ such that $\bar{v} \in N(\mathbb{Z}_{p^k})$.

According to **Theorem 1**, nilpotent vertices are adjacent to all other vertices in the graph $\Gamma_N(\mathbb{Z}_{p^k})$, as evident. Since the total number of vertices is p^k, the degree of a nilpotent vertex is $p^k - 1$

ii. For every $\bar{v} \in V\left(\Gamma_N(\mathbb{Z}_{p^k})\right)$ such that $\bar{v} \notin N(\mathbb{Z}_{p^k})$.

From **Theorem 1**, $\bar{v}$ is adjacent to all nilpotent vertices of the ring $\mathbb{Z}_{p^k}$. This means $\bar{v}$ is adjacent to $p^{(k-1)}$ vertices in the graph. However, according to Theorem 1, $\bar{v}$ is not adjacent to any non-nilpotent vertices, meaning it is adjacent only to nilpotent vertices. Thus, the degree of $\bar{v}$ is $p^{(k-1)}$. ■

4 Conclusion

This study explores the structural and numerical properties of nilpotent graphs associated with integer modulo rings of prime power order. By characterizing the nilpotent elements and analyzing their graph-theoretic relationships, we established key results related to connectivity, chromatic number, clique structure, and subgraph formations. These findings demonstrate how graph representations can offer valuable insights into ring-theoretic concepts.

However, this work is limited to specific types of rings, particularly those with prime power order, and focuses primarily on undirected graphs formed by multiplicative adjacency. Future studies may consider broader classes of rings, different adjacency conditions, or extend the analysis to weighted or directed graphs. Additionally, exploring algorithmic or computational approaches for large-scale structures remains an open challenge.

From a practical perspective, the insights gained from nilpotent graphs can contribute to developments in computational algebra, coding theory, and network design, where the structure of underlying algebraic systems plays a crucial role. Further research could also investigate connections with chemical graph theory or cryptographic applications, where ring-based structures are commonly used.

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